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Explicit monodromy of Moore–Read wavefunctions on a torus

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Abstract
We construct the wavefunctions for the Moore–Read $\nu = 5/2$ quantum Hall state on a torus in the presence of two quasiholes. These explicit wavefunctions allow us to compute the monodromy matrix that describes the effect of quasihole motion on the space of degenerate ground states. The result agrees with the recent discussion by Oshikawa *et al* (*Ann. Phys.* at press). Our calculation provides a conformal field theory explanation of why certain transitions between ground states are forbidden. It is because taking a quasihole around a generator of the torus can change the fusion channel of the two quasiholes, and this requires a change of parity of the electron number in some of the ground states.

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1. Introduction

The degeneracy, first noted in numerical work by Yoshioka *et al* and Su [1], of the fractional quantum Hall ground state on a torus is an essential feature of such states. In its absence, Laughlin's general gauge argument would require the Hall conductance to be an integer. Explicit wavefunctions for Laughlin states on a torus were written by Haldane and Rezayi [2], who identified the degeneracy as arising from translations of the centre of mass of the incompressible Hall fluid. Later, Wen and Niu realized [3] that the degeneracy persists even in the absence of translation symmetry and that the degree of degeneracy was sensitive to the global topology of the space in which the Hall fluid resides. This observation leads them to introduce the notion of *topological order* as a characterization of the strongly correlated ground state. Moore and Read [4] revealed the more general structure of topological order by pointing out its connection to rational conformal field theory. Since topological order is distinguished by the quantum numbers of ground state and excitations, a change of topological order requires a quantum phase transition corresponding to a substantial rearrangement of the many-body Hilbert space. Such phase transitions are not usually associated with symmetry breaking, and the resulting topological phases possess no conventional order parameters. They therefore fall outside the conventional Landau theory of phase transitions.

It is possible for a quantum fluid to have topologically degenerate states even when it lives on the plane. The degree of degeneracy then depends on the number and type of vortex-like defects in the fluid. Braiding these defects produces transitions between the topologically protected degenerate states, and it has been suggested that such manipulations may be exploited for quantum computation [5]. One such topological phase, the Moore–Read Pfaffian state, is the likely candidate for the observed $\nu = 5/2$ quantum Hall state [4]. The Pfaffian wavefunction is easy to write on the plane. When the fluid is placed on a torus, however, the wavefunction becomes more complicated. Counting the number of degenerate states is harder than counting the number states for a Laughlin fractional quantum Hall state, and the answer depends on whether the number of electrons in the system is odd or even. Nonetheless, the ground-state wavefunctions have been constructed [6–8]—although some issues still remain. Using general principles that do not require knowledge of the explicit Pfaffian wavefunctions, and by building on what is understood for Abelian quasiparticles [9], Oshikawa *et al* [10] have identified the topological quasiparticle operations that transform one Moore–Read ground state to another.

In this paper, we investigate how the topological operations of Oshikawa *et al*'s affect the many-electron wavefunctions. To do this, we first construct the Moore–Read Pfaffian states on a torus in the presence of two quasihole excitations. We then explicitly exhibit their transformation properties under topological operations on the quasihole positions. In this process, we provide a conformal field theory explanation of why the monodromy matrix acting on the space of the degenerate ground states should be block-diagonal with respect to the parity of the spin structure. It is because the spin structure dictates both the fusion channel of two quasiholes and the parity of the number of electrons.

2. Laughlin states

Before tackling the Pfaffian state, it helps to recall the properties of the simpler Laughlin states on a torus [2, 7, 11]. A torus can be regarded as a rectangle with sides L_x and L_y periodic boundary conditions. In the Landau gauge, with $\mathbf{A} = -By\hat{\mathbf{x}}$, the periodic boundary conditions on the wavefunction are twisted by a gauge transformation that is necessary to prepare a particle that leaves the top edge of the rectangle for its reappearance at the bottom edge. For a single electron we have

$$\psi(x + L_x, y) = \psi(x, y) \qquad \psi(x, y + L_y) = \exp(-\mathrm{i}L_y x/l^2)\psi(x, y), \tag{1}$$

where $l = \sqrt{\hbar/eB}$ is the magnetic length of the system. This twisted boundary condition must continue to hold, particle by particle, for the many-particle wavefunction. In our Landau gauge, the lowest Landau level many-particle wavefunction takes the form

$$\Psi(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_{N_e}) = \exp\left(-\sum_i y_i^2 \Big/ 2l^2\right) f(z_1, z_2, \ldots, z_{N_e}), \tag{2}$$

where $z_i = (x_i + \mathrm{i}y_i)/L_x$ and f is a holomorphic function in each of z_i. Combining equations (1) and (2) shows that f must satisfy the following quasi-periodicity conditions:

$$\begin{aligned} &f(z_1, \ldots, z_i + 1, \ldots) = f(z_1, \ldots, z_i, \ldots), \\ &f(z_1, \ldots, z_i + \tau, \ldots) = \exp[-\mathrm{i}\pi N_s(2z_i + \tau)] \cdot f(z_1, \ldots, z_i, \ldots). \end{aligned} \tag{3}$$

Here, $\tau = \mathrm{i}L_y/L_x$ and $N_s = L_x L_y/2\pi l^2$ is the number of flux quanta passing through the torus. (These conditions on f preserve their form for a torus which is a periodic parallelogram, rather than a rectangle. In this case τ is no longer purely imaginary, but should retain a positive imaginary part.)

By integrating $\mathrm{d}/\mathrm{d}z_i\left[\ln f\left(z_1, \ldots, z_{N_e}\right)\right]$ around the boundaries of the rectangle, and using equation (3) to combine the contributions of the opposite sides, we see that the number of zeros of f, considered as a function of z_i, is N_s. The precise locations of these N_s zeros will depend on the positions of the other z_j, but they are subject to a non-trivial constraint [11]. Suppose that $g(z)$ is meromorphic and doubly periodic:

$$g(z) = g(z+1) = g(z+\tau). \tag{4}$$

By evaluating the integral

$$I = \frac{1}{2\pi\mathrm{i}} \oint z \frac{g'(z)}{g(z)}\,\mathrm{d}z \tag{5}$$

around the edges of the period parallelogram $0 \to 1 \to 1+\tau \to \tau \to 0$, and again combining the contributions from opposite sides, we find that

$$I = m + n\tau \tag{6}$$

where m and n are integers. On the other hand, if the poles of $g(z)$ in the period parallelogram are at $z = b_i$ and the zeros at $z = a_i$, then $I = \sum_i a_i - \sum_i b_i$. Taken together with equation (6), this means that the sum of zeros minus the sum of poles vanishes modulo periods. This is *Abel's theorem* for the torus [12]. Although $f\left(z_1, \ldots, z_{N_e}\right)$ is it not doubly periodic, the ratio

$$g(z) \equiv f\left(z, , z_2 \ldots, z_{N_e}\right) \big/ f\left(z, z_2' \ldots, z_{N_e}'\right) \tag{7}$$

is a doubly periodic meromorphic function whose zeros are at the zeros of $f\left(z, z_2 \ldots, z_{N_e}\right)$ and whose poles are at the zeros of $f\left(z, z_2' \ldots, z_{N_e}'\right)$. Consequently, the sum of N_s zeros of $f\left(z_1, z_2, \ldots, z_{N_e}\right)$ considered as a function of z_1 is independent of other electron coordinates. The same is true of the zeros of f considered as a function of any of z_i.

The defining characteristic of Laughlin wavefunction at filling fraction $1/q$ (where q is an odd integer and $N_s = qN_e$) is that it vanishes as $(z_i - z_j)^q$ as any z_i approaches any other z_j. When combined with the condition discussed in the last paragraph, namely that the sum of all zeros for each z_i should be constant, the only possible form of the holomorphic part of the wavefunction for the Laughlin state on the torus is [2, 11]

$$f\left(z_1, z_2, \ldots, z_{N_e}\right) = F_{\mathrm{cm}}\left(\sum_i z_i\right) \prod_{i<j} [\vartheta_1(z_i - z_j)]^q. \tag{8}$$

Here $F_{\mathrm{cm}}(Z)$ is a holomorphic function possessing q zeros, and $\vartheta_1(z)$ is one of the four Jacobi theta functions:

$$\begin{aligned}
\vartheta_1(z) &= -\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}(z|\tau),\\
\vartheta_2(z) &= \vartheta\begin{bmatrix}1/2\\0\end{bmatrix}(z|\tau),\\
\vartheta_3(z) &= \vartheta\begin{bmatrix}0\\0\end{bmatrix}(z|\tau),\\
\vartheta_4(z) &= \vartheta\begin{bmatrix}0\\1/2\end{bmatrix}(z|\tau),
\end{aligned} \tag{9}$$

where the *theta function with characteristics* is defined as

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z|\tau) = \sum_{n=-\infty}^{\infty} \exp[\mathrm{i}\pi\tau(n+a)^2 + 2\pi\mathrm{i}(n+a)(z+b)]. \tag{10}$$

We will usually write the theta function with characteristics as $\vartheta[\alpha](z,\tau)$ where α is the vector $(a,b)^T$.

Each of the theta functions possesses a single zero in the period parallelogram, but note that for $a, b \in \mathbb{Z}/2$

$$\vartheta[\alpha](-z,\tau) = (-1)^{4ab}\vartheta[\alpha](z,\tau), \tag{11}$$

so only ϑ_1 obeys $\vartheta_1(-z) = -\vartheta_1(z)$ and has its zero at $z = 0$. All theta functions are quasi-periodic, and, in particular,

$$\begin{aligned} \vartheta_1(z+1) &= -\vartheta_1(z), \\ \vartheta_1(z+\tau) &= -\exp[-\mathrm{i}\pi(2z+\tau)]\vartheta_1(z). \end{aligned} \tag{12}$$

Equations (3), (8) and (12), together with $N_s = qN$, require the following quasi-periodicity conditions for $F_{\mathrm{cm}}(Z)$ [2]:

$$\begin{aligned} F_{\mathrm{cm}}(Z+1) &= (-1)^{(N_s-q)}F_{\mathrm{cm}}(Z), \\ F_{\mathrm{cm}}(Z+\tau) &= (-1)^{(N_s-q)}\exp[-\mathrm{i}\pi q(2z+\tau)]F_{\mathrm{cm}}(Z). \end{aligned} \tag{13}$$

A convenient basis for such $F_{\mathrm{cm}}(Z)$ is provided by the set of q functions [7, 13]:

$$F_{\mathrm{cm}}^{(m)}(Z) = \vartheta \begin{bmatrix} m/q + (N_s-q)/2q \\ -(N_s-q)/2 \end{bmatrix} (qZ|q\tau), \tag{14}$$

where m is an integer defined mod q. The resulting set of q linearly independent wavefunctions differ only by a rigid translation along the y-direction, and in the limit $N_e \to \infty$ each electron needs move only infinitesimal amount to cause this shift. The distinct wavefunctions are locally indistinguishable, and local perturbations change the energy of these states by same amount. The q-fold degeneracy is therefore unaffected by such local perturbations.

If now N_h quasiholes are inserted at w_i, then $N_s = qN_e + N_h$. Since f should now vanish to the first power as $z_i \to w_j$, the only way to have the sum of the zeros of each z_i independent of $\{w_j\}$ is to set [2, 11]

$$f^{(m)}(z_1, z_2, \ldots, z_{N_e}) = F_{\mathrm{cm}}^{(m)}\left(\sum_i z_i + \frac{\sum_j w_j}{q}\right) \prod_{i,j}\vartheta_1(z_i - w_j)\prod_{i<j}[\vartheta_1(z_i - z_j)]^q. \tag{15}$$

It is useful to include a purely $\{w_j\}$-dependent part to the wavefunction normalization so that

$$\begin{aligned} &\Psi^{(m)}(\mathbf{r}_1, \ldots, \mathbf{r}_{N_e}; \mathbf{R}_1, \ldots, \mathbf{R}_{N_h}) \\ &\quad = \exp\left(-\sum_i \frac{y_i^2}{2l^2}\right)\exp\left(-\sum_j \frac{\eta_j^2}{2ql^2}\right)F_{\mathrm{cm}}^{(m)}\left(\sum_i z_i + \frac{\sum_j w_j}{q}\right) \\ &\qquad \times \prod_{i<j}[\vartheta_1(w_i - w_j)]^{1/q}\prod_{i,j}\vartheta_1(z_i - w_j)\prod_{i<j}[\vartheta_1(z_i - z_j)]^q, \end{aligned} \tag{16}$$

where $w_i = (\xi_i + \mathrm{i}\eta_i)/L_x$ and $\mathbf{R} = (\xi, \eta)$. It was argued by Einarsson [14] that taking one quasihole around another on a torus should result in the same phase factor on a torus as on plane and this factor is accounted for by $\prod_{i<j}[\vartheta_1(w_i - w_j)]^{1/q}$. The exponent of the new Gaussian factor can be explained by the fact that the charge of a quasihole is e/q. These

added factors in equation (16) should therefore have converted all Berry phases into explicit monodromies.

One can easily check that for $\{z_i\}$ boundary conditions of equation (16) are exactly that of equation (1), provided that N_s of equation (13) is equal to $qN_e + N_h$. Under a straight-line analytic continuation $z \to z \pm 1$ we have

$$[\vartheta_1(z)]^{1/q} \to \exp(\pm\,\mathrm{i}\pi/q)[\vartheta_1(z)]^{1/q} \tag{17}$$

and as $z \to z \pm \tau$ we have

$$[\vartheta_1(z)]^{1/q} \to \exp(\mp\,\mathrm{i}\pi/q)\exp\left[-\frac{\mathrm{i}\pi}{q}(\pm 2z + \tau)\right][\vartheta_1(z)]^{1/q}. \tag{18}$$

Also note for the centre-of-mass wavefunction:

$$\begin{aligned} F_{\mathrm{cm}}^{(m)}(Z \pm 1/q) &= (-1)^{N_e}\,\mathrm{e}^{\pm\pi\mathrm{i}(N_h - q)/q}\exp\left[\pm 2\pi\mathrm{i}\frac{m}{q}\right]F_{\mathrm{cm}}^{(m)}(Z), \\ F_{\mathrm{cm}}^{(m)}(Z \pm \tau/q) &= (-1)^{N_e}\,\mathrm{e}^{\pm\pi\mathrm{i}(N_h - q)/q}\exp[-\pi\mathrm{i}(\pm 2z + \tau/q)]F_{\mathrm{cm}}^{(m\pm 1)}(Z). \end{aligned} \tag{19}$$

Taking a quasihole around one of the generators of the torus therefore affects the following transformations: under $\mathbf{R}_i \to \mathbf{R}_i \pm L_x\hat{\mathbf{x}}$ we have

$$\begin{aligned} \Psi^{(m)}\big(\mathbf{r}_1, \ldots, \mathbf{r}_{N_e}; \mathbf{R}_1, \ldots, \mathbf{R}_i, \ldots\big) &\to \exp\left[\pm 2\pi\mathrm{i}\left(\frac{m}{q} + \frac{2N_h - q - 1}{2q}\right)\right] \\ &\quad\times \Psi^{(m)}(\mathbf{r}_1, \ldots, \mathbf{r}_{N_e}; \mathbf{R}_1, \ldots, \mathbf{R}_i, \ldots), \end{aligned} \tag{20}$$

and under $\mathbf{R}_i \to \mathbf{R}_i \pm L_y\hat{\mathbf{y}}$ we have

$$\begin{aligned} \Psi^{(m)}\big(\mathbf{r}_1, \ldots, \mathbf{r}_{N_e}; \mathbf{R}_1, \ldots, \mathbf{R}_i, \ldots\big) &\to -\mathrm{e}^{\pm\mathrm{i}\pi/q}\exp(\mp\,\mathrm{i}L_y\xi_i/ql^2) \\ &\quad\times \Psi^{(m\pm 1)}\big(\mathbf{r}_1, \ldots, \mathbf{r}_{N_e}; \mathbf{R}_1, \ldots, \mathbf{R}_i, \ldots\big). \end{aligned} \tag{21}$$

Equations (20), (21) reproduce the operator transformation of Wen and Niu [3] up to a gauge transformation: motion of the quasihole about one torus generator reproduces the state up to phase, and motion about the other generator rolls the ground state over into another one[1]. These operations and their effects are analogous to the operations appearing in the Verlinde algebra of conformal field theory.

3. Moore–Read state

The holomorphic part of the $\nu = 1/2$ Moore–Read state wavefunction on the plane is

$$f_{\mathrm{MR}}(z_1, \ldots, z_{2n}) = \mathrm{Pf}\left(\frac{1}{z_i - z_j}\right)\prod_{i<j}(z_i - z_j)^2, \tag{22}$$

where the Pfaffian of a $2n$-by-$2n$ antisymmetric matrix is defined as

$$\mathrm{Pf}(A) = \frac{1}{2^n n!}\sum_{P\in S_{2n}}\mathrm{sgn}(P)\prod_{i=1}^{n}A_{P(2i-1),P(2i)}. \tag{23}$$

Here P runs over all permutation of $2n$ objects. This wavefunction is an antisymmetric polynomial in z_i, the poles in the Pfaffian part having cancelled against some of the zeros in the Laughlin–Jastrow factor. The suppression of these zeros by the Pfaffian factor increases the amplitude for the one electron to approach another. The Pfaffian can therefore be considered to

[1] The constant factor $-\exp(\pm\,\mathrm{i}\pi/q)$ can be taken care of if the ground-state wavefunction is redefined with suitable phase factor.

indicate a pairing between electrons [6], and this pairing is closely related to the BCS pairing in $p + \mathrm{i}p$ superconductors [8].

In constructing the Moore–Read wavefunctions on a torus we must satisfy the boundary conditions (3). These boundary conditions come from the gauge choice and the fact that all electrons are in the lowest Landau level; they do not depend on what correlated many-body state the electrons are in.

Now the Pfaffian part is equal to the correlator of $2n$ chiral Majorana fermion fields of the critical Ising model

$$\langle\psi(z_1)\cdots\psi(z_{2n})\rangle = \mathrm{Pf}\left(\frac{1}{z_i - z_j}\right). \tag{24}$$

There is more than one way to put such a correlator on a torus. This is because we are free to give the Ising fields periodic or antiperiodic boundary conditions around each generator. We will see that there is an intricate interplay between the boundary conditions (3) required of the physical electrons and the boundary condition choices we make for the Ising ψ's.

3.1. Even-spin-structure sector

The Laughlin–Jastrow part of the wavefunction on the torus can be dealt with by the same $(z_i - z_j) \to \vartheta_1(z_i - z_j)$ substitution as before. The $1/(z_i - z_j)$ factors in the Pfaffian require a more careful treatment. Because the Pfaffian contains *sums* of products any z-dependent factors arising from the $z \to z+1$ or $z \to z+\tau$ quasi-periodicity properties will not appear as an overall common factor. Consequently, we should arrange that there are *no* z-dependent factors at all, and simply setting $1/(z_i - z_j) \to 1/\vartheta_1(z_i - z_j)$ will not do. A possible choice is to set [6]

$$\frac{1}{z_i - z_j} \to \frac{\vartheta[\alpha](z_i - z_j)}{\vartheta_1(z_i - z_j)} \tag{25}$$

for some choice of $\alpha = (1/2, 0)^T, (0, 0)^T, (0, 1/2)^T$. The functions $\vartheta[\alpha](z)/\vartheta_1(z)$ now obey the following boundary conditions:

$$\begin{aligned}\frac{\vartheta[\alpha](z+1)}{\vartheta_1(z+1)} &= -\mathrm{e}^{2\pi\mathrm{i}a}\frac{\vartheta[\alpha](z)}{\vartheta_1(z)},\\ \frac{\vartheta[\alpha](z+\tau)}{\vartheta_1(z+\tau)} &= -\mathrm{e}^{2\pi\mathrm{i}b}\frac{\vartheta[\alpha](z)}{\vartheta_1(z)}.\end{aligned} \tag{26}$$

These functions are proportional to the two-point functions

$$\langle\psi(z)\psi(z')\rangle_\alpha = \frac{\vartheta_1'(0)\vartheta[\alpha](z-z')}{\vartheta[\alpha](0)\vartheta_1(z-z')}, \tag{27}$$

for chiral Majorana fermions with different spin structures, i.e. with different periodic or antiperiodic boundary conditions round the generators of the torus. At least one of these boundary conditions must be antiperiodic. If this condition is not met, the fermion propagator has a zero mode, and the usual two-point function does not exist. We are therefore at the moment considering only *even*-spin structures, i.e. spin structures $\alpha = (a, b)^T$ having $4ab$ an even integer. It should be noted once again that these periodic and antiperiodic boundary conditions are merely properties of ingredients in the Pfaffian. We are *not* changing the boundary conditions of the many-body wavefunction. Instead the sign factors are accommodated by the centre-of-mass wavefunction. This means that the centre-of-mass wavefunctions obtained by setting $q = 2$ in equation (14),

$$\tilde{F}^{(m)}_{\mathrm{cm}}(Z) = \vartheta\begin{bmatrix} m/2 + (N_s - 2)/4 \\ -(N_s - 2)/2 \end{bmatrix}(2Z|2\tau), \tag{28}$$

will no longer suffice. Instead there are distinct centre-of-mass wavefunctions for each $\alpha = (a, b)^T$:

$$F_{\rm cm}^{(\alpha,m)}(Z) = \vartheta \begin{bmatrix} m/2 + (N_s - 2)/4 + (1-2a)/4 \\ -(N_s-2)/2 - (1-2b)/2 \end{bmatrix} (2Z|2\tau). \tag{29}$$

Here m is an integer defined mod 2. (It will be shown later that equation (28) *is* the centre-of-mass wavefunction in the *odd*-spin structure.) The complete wavefunction

$$\Psi^{(\alpha,m)}(\mathbf{r}_1,\ldots,\mathbf{r}_{N_e}) = \exp\left(-\sum_i y_i^2 \Big/ 2l^2\right) F_{\rm cm}^{(\alpha,m)}\left(\sum_i z_i\right) \times \mathrm{Pf}\left(\frac{\vartheta[\alpha](z_i - z_j)}{\vartheta_1(z_i - z_j)}\right) \prod_{i<j} [\vartheta_1(z_i - z_j)]^2, \tag{30}$$

with $\alpha = (1/2, 0)^T$, $(0, 0)^T$ or $(0, 1/2)^T$, now satisfies the boundary conditions (3). Note that the additional α-dependent terms that equation (29) possesses in comparison with equation (28) ensures that that the boundary conditions of the total wavefunctions are same for different α's. The three spin-structure choices coupled with the two possible values of m in equation (30) give us the six-fold degeneracy of the even-spin-structure Moore–Read state on the torus [6, 8, 10].

Now we consider inserting quasiholes. A charge $e/2$ quasihole excitation in one of these degenerate ground states is little different from a quasihole in a Laughlin state. Such a quasihole has one quantum of flux and consequently $N_s = 2N_e + 1$. When an $e/2$ quasihole is inserted at w, the holomorphic part of the wavefunction becomes

$$f^{(\alpha,m)}(z_1,\ldots,z_{N_e}; w) = F_{\rm cm}^{(\alpha,m)}\left(\sum_i z_i + \frac{w}{2}\right) \times \mathrm{Pf}\left(\frac{\vartheta[\alpha](z_i - z_j)}{\vartheta_1(z_i - z_j)}\right) \prod_i \vartheta_1(z_i - w) \prod_{i<j} [\vartheta_1(z_i - z_j)]^2. \tag{31}$$

When this quasihole is carried around the generators, the wavefunction equation (31) behaves almost the same as the torus Laughlin wavefunction. Under $w \to w \pm 1$ we have

$$f^{(\alpha,m)}(z_1,\ldots,z_{N_e}; w) \to \begin{cases} (-1)^m f^{(\alpha,m)}(z_1,\ldots,z_{N_e}; w), & \alpha = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} \\ \mp \mathrm{i}(-1)^m f^{(\alpha,m)}(z_1,\ldots,z_{N_e}; w), & \alpha = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} \end{cases} \tag{32}$$

and under $w \to w \pm \tau$ we have

$$f^{(\alpha,m)}(z_1,\ldots,z_{N_e}; w) \to \begin{cases} \exp\left[-\frac{\mathrm{i}\pi N_s}{2}(\pm 2w + \tau)\right] f^{(\alpha,m+1)}(z_1,\ldots,z_{N_e}; w), & \alpha = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \mp \mathrm{i} \exp\left[-\frac{\mathrm{i}\pi N_s}{2}(\pm 2w + \tau)\right] f^{(\alpha,m+1)}(z_1,\ldots,z_{N_e}; w), & \alpha = \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}. \end{cases} \tag{33}$$

This is to be expected because creating the $e/2$ quasihole breaks no pairs and so has no effect on the BCS pairing characterizing the Moore–Read state. (These transformations have been discussed by Oshikawa *et al* in the operator language [10]. There, the monodromy was associated with the adiabatic insertion of a unit flux quantum into the 'holes' of the torus.)

The $e/2$ quasihole is not, however, the elementary excitation for the Moore–Read state. By allowing pair-breaking, a charge $e/2$ quasihole with one quantum flux can fractionalize into two charge $e/4$ quasiholes, each possessing a *half* quantum of flux [4, 8]. To construct a wavefunction with two charge $e/4$ quasiholes, the Pfaffian part needs to be modified, as this is the part that describes the BCS pairing.

There are three basic conditions to be considered when writing such a two-quasihole wavefunction. The first is that when these two quasiholes are brought together we should recover equation (31). The second condition is that if two *paired* electrons wind around a single charge $e/4$ quasihole, it should result in accumulation of phase 2π. The third condition is that, since the argument of the centre-of-mass wavefunction becomes $\sum_i z_i + (w_1 + w_2)/4$, the phase factor that multiplies the Pfaffian when we take $z_i \to z_i + \tau$ should change to accommodate this. Greiter *et al* obtained wavefunctions satisfying these three conditions [6]. The holomorphic part of their wavefunction is

$$f^{(\alpha,m)}(z_1, \ldots, z_{N_e}; w_1, w_2) = F_{\mathrm{cm}}^{(\alpha,m)}\left(\sum_i z_i + \frac{w_1 + w_2}{4}\right) \mathrm{Pf}\left(M_{ij}^{\alpha}\right) \prod_{i<j} [\vartheta_1(z_i - z_j)]^2, \quad (34)$$

with

$$M_{ij}^{\alpha} = \frac{\vartheta[\alpha](z_i - z_j + w_{12}/2)\vartheta_1(z_i - w_1)\vartheta_1(z_j - w_2) + (i \leftrightarrow j)}{2\vartheta_1(z_i - z_j)}, \quad (35)$$

and $w_{12} = w_1 - w_2$. (Note that $(i \leftrightarrow j)$ refers to a term that differs from the previous term only by the exchange of the index i and j; hence, for equation (35), $(i \leftrightarrow j) \equiv \vartheta[\alpha](z_j - z_i + w_{12}/2)\vartheta_1(z_j - w_1)\vartheta_1(z_i - w_2)$.) As far as translation of electrons around the torus generators are concerned equation (34) has same transformation as equation (31) or as the holomorphic part of equation (30).

The above considerations are *not*, however, sufficient for obtaining the complete two-quasihole wavefunctions. Equations (34) and (35) satisfy constraints involving only the electron coordinates z_i. There are also important factors that depend only on w_1 and w_2. To obtain the complete dependence on the quasihole coordinates as well, we need to calculate the holomorphic conformal blocks of the correlators for critical Ising model.

The starting point is to observe from Moore and Read's original derivation how the Pfaffian part of the Moore–Read state wavefunction with two quasiholes is obtained when placed on the plane [4]:

$$\mathrm{Pf}\left(\frac{(z_i - w_1)(z_j - w_2) + (i \leftrightarrow j)}{z_i - z_j}\right) = \left\langle \psi(z_1)\cdots\psi(z_{N_e})\sigma(w_1)\sigma(w_2)\right\rangle \times w_{12}^{1/8} \prod_i [(z_i - w_1)(z_i - w_2)]^{1/2}, \quad (36)$$

where ψ is the Majorana fermion and σ is the spin field. One can immediately see that the Pfaffian part of the wavefunction on a torus would be equal to

$$\left\langle \psi(z_1)\cdots\psi(z_{N_e})\sigma(w_1)\sigma(w_2)\right\rangle_{\alpha} [\vartheta_1(w_{12})]^{1/8} \prod_i [\vartheta_1(z_i - w_1)\vartheta_1(z_i - w_2)]^{1/2}. \quad (37)$$

Here $\alpha = (a, b)^T$ denotes the boundary conditions, equation (26), of the ψ field on the torus, just as in the case of equation (27).

The following Ising model correlators have been obtained in the even-spin structures on the torus by Di Francesco *et al* [15, 16]:

$$\frac{\langle\psi(z_i)\psi(z_j)\sigma(w_1)\sigma(w_2)\rangle_\alpha}{\langle\sigma(w_1)\sigma(w_2)\rangle_\alpha} = \frac{\vartheta_1'(0)}{2\vartheta_1(z_i - z_j)} \times \left[\frac{\vartheta[\alpha](z_i - z_j + w_{12}/2)}{\vartheta[\alpha](w_{12}/2)}\left(\frac{\vartheta_1(z_i - w_1)\vartheta_1(z_j - w_2)}{\vartheta_1(z_i - w_2)\vartheta_1(z_j - w_1)}\right)^{1/2} + (i \leftrightarrow j)\right],$$

$$\langle\sigma(w_1)\sigma(w_2)\rangle_\alpha = \left(\frac{\vartheta[\alpha](w_{12}/2)}{\vartheta[\alpha](0)}\right)^{1/2}\left(\frac{\vartheta_1'(0)}{\vartheta_1(w_{12})}\right)^{1/8}. \tag{38}$$

(To be precise, these are the chiral holomorphic parts extracted from the non-chiral Ising field correlators obtained by Di Francesco *et al*.) From the antisymmetry of the ψ field under exchange, and from the conditions

$$\lim_{z_1\to z_2}(z_1 - z_2)\langle\psi(z_1)\psi(z_2)\psi(z_3)\cdots\psi(z_{N_e})\sigma(w_1)\sigma(w_2)\rangle_\alpha = \langle\psi(z_3)\cdots\psi(z_{N_e})\sigma(w_1)\sigma(w_2)\rangle_\alpha,$$

$$\lim_{z_i\to z_j}(z_i - z_j)\frac{\langle\psi(z_i)\psi(z_j)\sigma(w_1)\sigma(w_2)\rangle_\alpha}{\langle\sigma(w_1)\sigma(w_2)\rangle_\alpha} = 1, \tag{39}$$

we can obtain the N_e-point ψ field correlator (N_e even) as

$$\langle\psi(z_1)\cdots\psi(z_{N_e})\sigma(w_1)\sigma(w_2)\rangle_\alpha = \langle\sigma(w_1)\sigma(w_2)\rangle_\alpha \,\mathrm{Pf}\left(\frac{\langle\psi(z_i)\psi(z_j)\sigma(w_1)\sigma(w_2)\rangle_\alpha}{\langle\sigma(w_1)\sigma(w_2)\rangle_\alpha}\right). \tag{40}$$

Equations (37), (38) and (40) tell us that the wavefunctions of equation (34) need to be modified in the following way if they are to have correct dependence on quasihole coordinates:

$$f^{(\alpha,m)}(z_1,\ldots,z_{N_e};w_1,w_2) = F_{\mathrm{cm}}^{(\alpha,m)}\left(\sum_i z_i + \frac{w_1+w_2}{4}\right)[\vartheta[\alpha](w_{12}/2)]^{1/2}\,\mathrm{Pf}\left(\tilde{M}_{ij}^\alpha\right) \times \prod_{i<j}[\vartheta_1(z_i - z_j)]^2, \tag{41}$$

with

$$\tilde{M}_{ij}^\alpha = \frac{\vartheta[\alpha](z_i - z_j + w_{12}/2)\vartheta_1(z_i - w_1)\vartheta_1(z_j - w_2) + (i \leftrightarrow j)}{2\vartheta_1(z_i - z_j)\vartheta[\alpha](w_{12}/2)}. \tag{42}$$

Note that while equation (38) is *not* single-valued in coordinates of the ψ fields—taking a ψ around a σ results in a minus sign—the wavefunction equation (41) *is* single-valued in the electron coordinate. This is because the $[\vartheta_1(w_{12})]^{1/8}\prod_i[\vartheta_1(z_i - w_1)\vartheta_1(z_i - w_2)]^{1/2}$ factor (in addition to the Laughlin–Jastrow factor) makes everything analytic in equation (41), same for $[\vartheta[\alpha](w_{12}/2)]^{1/2}$. Thus, the wavefunction is single-valued in electron coordinate and has no singularity. Note that constants $\vartheta[\alpha](0)$ and $\vartheta_1'(0)$ have been ignored in equation (41). (This means in the limit $w_1 \to w_2$ equation (41) will differ from equation (31) by some multiplicative constant.)

We now ask what happens when one of the quasiholes is translated around the generators. In contrast to equation (33) where only m changed, this operation will result in a change in α. To see what happens to the wavefunctions, we require the following standard theta function identities [12, 15]:

$$\begin{aligned}
\vartheta_2(z \pm 1/2) &= \mp\vartheta_1(z),\\
\vartheta_2(z \pm \tau/2) &= \exp[-\mathrm{i}\pi(\pm z + \tau/4)]\vartheta_3(z),\\
\vartheta_3(z \pm 1/2) &= \vartheta_4(z),\\
\vartheta_3(z \pm \tau/2) &= \exp[-\mathrm{i}\pi(\pm z + \tau/4)]\vartheta_2(z),\\
\vartheta_4(z \pm 1/2) &= \vartheta_3(z),\\
\vartheta_4(z \pm \tau/2) &= \pm\mathrm{i}\exp[-\mathrm{i}\pi(\pm z + \tau/4)]\vartheta_1(z),
\end{aligned} \tag{43}$$

together with the centre-of-mass wavefunction formulae,

$$\begin{aligned}
&F_{\rm cm}^{(a=2,m)}(z\pm 1/4) = \exp(-{\rm i}(1/2\mp 1/2)\pi(m+N_s/2-1))\tilde{F}_{\rm cm}^{(m)}(z),\\
&F_{\rm cm}^{(a=2,m)}(z\pm \tau/4) = {\rm e}^{\pm {\rm i}\pi(N_s-1)/4}\exp[-{\rm i}\pi(\pm z+\tau/8)]F_{\rm cm}^{(a=3,m+1/2\mp 1/2)}(z),\\
&F_{\rm cm}^{(a=3,m)}(z\pm 1/4) = \exp(-{\rm i}(1/2\mp 1/2)\pi(m+N_s/2-1/2))F_{\rm cm}^{(a=4,m)}(z),\\
&F_{\rm cm}^{(a=3,m)}(z\pm \tau/4) = {\rm e}^{\pm {\rm i}\pi(N_s-1)/4}\exp[-{\rm i}\pi(\pm z+\tau/8)]F_{\rm cm}^{(a=2,m+1/2\pm 1/2)}(z),\\
&F_{\rm cm}^{(a=4,m)}(z\pm 1/4) = \exp({\rm i}(1/2\pm 1/2)\pi(m+N_s/2-1/2))F_{\rm cm}^{(a=3,m)}(z),\\
&F_{\rm cm}^{(a=4,m)}(z\pm \tau/4) = {\rm e}^{\pm {\rm i}\pi(N_s-2)/4}\exp[-{\rm i}\pi(\pm z+\tau/8)]\tilde{F}_{\rm cm}^{(m+1/2\pm 1/2)}(z).
\end{aligned} \tag{44}$$

In these centre-of-mass wavefunction formulae, the notation of equation (9) is used; that is, $a = 2$ stands for $\alpha = (1/2, 0)^T$, $a = 3$ for $\alpha = (0, 0)^T$ and $a = 4$ for $\alpha = (0, 1/2)^T$.

Note that the transformations of equation (43) are covariant with those of equation (44).

Consider the results not involving ϑ_1 or $\tilde{F}_{\rm cm}^{(m)}$:

$$\begin{aligned}
w_1 \to w_1\pm\tau :\ & f^{(a=2,m)}(z_1,\ldots,z_{N_e}; w_1, w_2) \to \exp[-{\rm i}\pi N_s(\pm w_1/2+\tau/4)]\\
&\times f^{(a=3,m+1/2\mp 1/2)}(z_1,\ldots,z_{N_e}; w_1, w_2),\\
w_1 \to w_1\pm 1 :\ & f^{(a=3,m)}(z_1,\ldots,z_{N_e}; w_1, w_2) \to (-1)^{N_e/2}\,{\rm e}^{{\rm i}\pi m(1/2\mp 1/2)}\\
&\times f^{(a=4,m)}(z_1,\ldots,z_{N_e}; w_1, w_2),\\
w_1 \to w_1\pm\tau :\ & f^{(a=3,m)}(z_1,\ldots,z_{N_e}; w_1, w_2) \to \exp[-{\rm i}\pi N_s(\pm w_1/2+\tau/4)]\\
&\times f^{(a=2,m+1/2\pm 1/2)}(z_1,\ldots,z_{N_e}; w_1, w_2),\\
w_1 \to w_1\pm 1 :\ & f^{(a=4,m)}(z_1,\ldots,z_{N_e}; w_1, w_2) \to (-1)^{N_e/2}\,{\rm e}^{{\rm i}\pi m(1/2\pm 1/2)}\\
&\times f^{(a=3,m)}(z_1,\ldots,z_{N_e}; w_1, w_2).
\end{aligned} \tag{45}$$

In Moore and Read's original wavefunction formulation [4] the Gaussian factor comes entirely determined from the charge sector. Since the charge of a quasihole considered here is $e/4$, an analogy with equation (16) shows that the total wavefunction should be

$$\begin{aligned}
\Psi^{(\alpha,m)}(\mathbf{r}_1,\ldots,\mathbf{r}_{N_e}; \mathbf{R}_1, \mathbf{R}_2) &= \exp\left(-\sum_i y_i^2\Big/2l^2\right)\\
&\times \exp\left[-\left(\eta_1^2+\eta_2^2\right)/8l^2\right] f^{(\alpha,m)}(z_1,\ldots,z_{N_e}; w_1, w_2).
\end{aligned} \tag{46}$$

Equations (45) and (46) give the following transformation rules for the total wavefunctions:

$$\begin{aligned}
\mathbf{R}_1 \to \mathbf{R}_1 \pm L_y\hat{\mathbf{y}} :\ & \Psi^{(a=2,m)}(\mathbf{r}_1,\ldots,\mathbf{r}_{N_e}; \mathbf{R}_1, \mathbf{R}_2) \to \exp(\mp {\rm i}L_y\xi_1/4l^2)\\
&\times \Psi^{(a=3,m+1/2\mp 1/2)}(\mathbf{r}_1,\ldots,\mathbf{r}_{N_e}; \mathbf{R}_1, \mathbf{R}_2),\\
\mathbf{R}_1 \to \mathbf{R}_1 \pm L_x\hat{\mathbf{x}} :\ & \Psi^{(a=3,m)}(\mathbf{r}_1,\ldots,\mathbf{r}_{N_e}; \mathbf{R}_1, \mathbf{R}_2) \to (-1)^{N_e/2}\,{\rm e}^{{\rm i}\pi m(1/2\mp 1/2)}\\
&\times \Psi^{(a=4,m)}(\mathbf{r}_1,\ldots,\mathbf{r}_{N_e}; \mathbf{R}_1, \mathbf{R}_2),\\
\mathbf{R}_1 \to \mathbf{R}_1 \pm L_y\hat{\mathbf{y}} :\ & \Psi^{(a=3,m)}(\mathbf{r}_1,\ldots,\mathbf{r}_{N_e}; \mathbf{R}_1, \mathbf{R}_2) \to \exp(\mp {\rm i}L_y\xi_1/4l^2)\\
&\times \Psi^{(a=2,m+1/2\pm 1/2)}(\mathbf{r}_1,\ldots,\mathbf{r}_{N_e}; \mathbf{R}_1, \mathbf{R}_2),\\
\mathbf{R}_1 \to \mathbf{R}_1 \pm L_x\hat{\mathbf{x}} :\ & \Psi^{(a=4,m)}(\mathbf{r}_1,\ldots,\mathbf{r}_{N_e}; \mathbf{R}_1, \mathbf{R}_2) = (-1)^{N_e/2}\,{\rm e}^{{\rm i}\pi m(1/2\pm 1/2)}\\
&\times \Psi^{(a=3,m)}(\mathbf{r}_1,\ldots,\mathbf{r}_{N_e}; \mathbf{R}_1, \mathbf{R}_2).
\end{aligned} \tag{47}$$

We see that, unlike for the Laughlin states, these states are not eigenstates with respect to $w_j \to w_j \pm 1$ transformations. However, the feature that this transformation does not change m persists. For the translations $w_j \to w_j \pm \tau$, there is a phase factor due to the gauge transformation $\mathbf{A} \to \mathbf{A} - BL_y\hat{\mathbf{x}}$. Comparing the gauge transformation phase factors of equation (21) to that of equation (47) confirms that the quasiholes in equation (47) have charge $e/4$.

Equation (45) left the following two cases unmentioned:

$$
\begin{aligned}
w_1 \to w_1 \pm 1 : f^{(a=2,m)}(z_1,\ldots,z_{N_e};w_1,w_2) &\to (-1)^{N_e/2}\,\mathrm{e}^{-\mathrm{i}\pi(1/4\mp 1/4)}\,\mathrm{e}^{\mathrm{i}\pi(m-1)(1/2\mp 1/2)} \\
&\quad\times \tilde{f}^{(m)}(z_1,\ldots,z_{N_e};w_1,w_2), \\
w_1 \to w_1 \pm \tau : f^{(a=4,m)}(z_1,\ldots,z_{N_e};w_1,w_2) &\to \mathrm{e}^{\mp\mathrm{i}\pi/4}\exp[-\mathrm{i}\pi N_s(\pm w_1/2+\tau/4)] \\
&\quad\times \tilde{f}^{(m+1/2\pm 1/2)}(z_1,\ldots,z_{N_e};w_1,w_2),
\end{aligned}
\tag{48}
$$

where

$$
\begin{aligned}
\tilde{f}^{(m)}(z_1,\ldots,z_{N_e};w_1,w_2) &= \tilde{F}^{(m)}_{\mathrm{cm}}\left(\sum_i z_i + \frac{w_1+w_2}{4}\right)[\vartheta_1(w_{12}/2)]^{1/2}\,\mathrm{Pf}\left(-\tilde{M}^{\alpha=(1/2,1/2)}_{ij}\right) \\
&\quad\times \prod_{i<j}[\vartheta_1(z_i-z_j)]^2.
\end{aligned}
\tag{49}
$$

The tilde mark is placed above f in equation (49) because, in some sense, it does *not* qualify as a Moore–Read state wavefunction. When the two quasiholes are merged in equation (49), the wavefunction simply vanishes; one does not get one of the wavefunctions of equation (31). This does not mean that equation (49) necessarily gives higher energy than equation (41). However, note that equation (31) differs from the ground states only by one quantum flux; equation (49) does not have a corresponding ground state in this sense, hence its disqualification. One can further regard having two $e/4$ quasiholes as differing from having a quasihole–quasiparticle pair merely by one quantum flux. This leads to the conclusion that if one creates a quasihole–quasiparticle pair out of one of $a = 2$ (or $a = 4$) ground states, translate the quasihole around the generator in the $x-$ (or y-) direction, and annihilate the pair, one does *not* return to a ground state; this means that the monodromy process can actually excite the system.

To find out what kind of excitation do we have here, we need to examine the vanishing of equation (49) as two quasiholes are brought together. An examination into equation (42) shows that the vanishing is not due to the Pfaffian part of equation (49). Indeed,

$$
\begin{aligned}
\lim_{w_2\to w_1}\tilde{M}^{\alpha=(1/2,1/2)}_{ij} &= -\frac{\vartheta_1(z_i-w_1)\vartheta_1(z_j-w_1)}{\vartheta_1'(0)} \\
&\quad\times\left[\frac{\vartheta_1'(z_i-z_j)}{\vartheta_1(z_i-z_j)} - \left(\frac{\vartheta_1'(z_i-w_1)}{\vartheta_1(z_i-w_1)} - \frac{\vartheta_1'(z_j-w_1)}{\vartheta_1(z_j-w_1)}\right)\right].
\end{aligned}
\tag{50}
$$

The vanishing of equation (49) is therefore solely due to $[\vartheta_1(w_{12}/2)]^{1/2}$. This term originates from the correlator of two quasiholes in the $\alpha = (1/2, 1/2)^T$ sector. This is the sector with the periodic boundary conditions around both generators for ψ fields:

$$
\langle\sigma(w_1)\sigma(w_2)\rangle_{\alpha=(1/2,1/2)}[\vartheta_1(w_{12})]^{1/8} \propto [\vartheta_1(w_{12}/2)]^{1/2}. \tag{51}
$$

The spin structure is therefore correlated with the internal state (fusion channel) of the two quasiholes. Since for $w \to 0$

$$
\langle\sigma(w)\sigma(0)\rangle_{\alpha=(1/2,1/2)} \sim w^{3/8}, \tag{52}
$$

whereas for $\alpha = (1/2, 0)^T$, $(0, 0)^T$, and $(0, 1/2)^T$

$$
\langle\sigma(w)\sigma(0)\rangle_{\alpha} \sim w^{-1/8}, \tag{53}
$$

the chiral Ising model operator product expansion

$$
\sigma(w)\sigma(0) \sim \frac{\mathbb{I}}{w^{1/8}} + \mathrm{const}\, w^{3/8}\psi(0) \tag{54}
$$

tells us that for $\alpha = (1/2, 1/2)^T$ two σ's fuse into ψ, whereas for $\alpha = (1/2, 0)^T$, $(0, 0)^T$, and $(0, 1/2)^T$ they fuse into $\mathbb{I}$. We see that our wavefunction approach makes manifest the observation of Oshikawa *et al* that, after the translation of a quasihole around a generator, the system may refuse to go back into a ground state because of a change in the $\sigma \times \sigma$ fusion channel [10]. This change was explained by Oshikawa *et al* by using the branch cut argument of Ivanov [17] and Stern *et al* [18]

From the preceding arguments, one can explain this change of the fusion channel in terms of the physical fusion in the manner discussed by Stone and Chung for the two-dimensional $p + \mathrm{i}p$ superconductor [19]. In the ground states, two σ's fused into $\mathbb{I}$ and the total fermion number on torus was even, enabling all fermions to be paired up. However, once the translation of one quasihole rolls the system over to the $\alpha = (1/2, 1/2)^T$ sector, σ's fused into a ψ, which, due to the conservation of total fermion number, is possible only if there is depairing of one of fermion pairs. Therefore, with this change of fusion channel, the system is left in an excited state.

This superconductor analogy also indicates that the parity of electron number in the ground state of the $\alpha = (1/2, 1/2)^T$ sector should be different from that of other sector. In this picture, after the roll over to the $\alpha = (1/2, 1/2)^T$ sector, there is a Bogoliubov quasiparticle excitation in the system. Now if superconductor with one Bogoliubov quasiparticle excitation has N electrons, this is equivalent to having a superposition of one hole excitation on a ground state with $N + 1$ electrons and one particle excitation on a ground state with $N - 1$ electrons. Thus, the change of fusion channel has to be accompanied by the change in the parity of the electron number in the ground state.

As previously commented, in discussing topological features, creating a quasiparticle–quasihole pair is equivalent to splitting a charge $e/2$ quasihole into two $e/4$ quasiholes; the only difference between two cases is that there is one more flux quantum for the latter case. In their paper, Oshikawa *et al* defined $\mathcal{T}_{x,y}$ to be a process in which a quasiparticle–quasihole pair is created and then the quasiparticle is dragged around the generator of the torus in the x- (or y-) direction to wrap around the system before it is pair-annihilated with the quasihole [10]. So one can make following correspondences between the analytic continuation of theta functions considered in this paper and the processes defined by Oshikawa *et al*:

$$w_1 \to w_1 + 1 \Leftrightarrow \mathcal{T}_x, \qquad w_1 \to w_1 + \tau \Leftrightarrow \mathcal{T}_y. \tag{55}$$

It will be shown in the appendices that the ground states labelled by a and m in this paper are eigenstates of $\mathcal{T}_x^{-2}$ and $\mathcal{T}_y^4$. Labelling eigenvalues for these operators divided by the gauge transformation factor as f_y and f'_x respectively, we obtain the following correspondences between the wavefunctions of this paper and the state vectors of Oshikawa *et al*:

$$\begin{aligned}
&\Psi^{(a=2,m=0)} \leftrightarrow |f_y = i, f'_x = 1\rangle && \Psi^{(a=2,m=1)} \leftrightarrow |f_y = -\mathrm{i}, f'_x = 1\rangle,\\
&\Psi^{(a=3,m=0)} \leftrightarrow |f_y = 1, f'_x = 1\rangle && \Psi^{(a=3,m=1)} \leftrightarrow |f_y = -1, f'_x = 1\rangle,\\
&(-1)^{N_e/2}\Psi^{(a=4,m=0)} \leftrightarrow |f_y = 1, f'_x = -1\rangle && (-1)^{N_e/2}\Psi^{(a=4,m=1)} \leftrightarrow |f_y = -1, f'_x = -1\rangle.
\end{aligned} \tag{56}$$

Adapting the continuation formulae of equation (48) to the correspondence made in equation (56), we find the following topological actions:

$$\begin{aligned}
&|f_y = 1, f'_x = 1\rangle && \mathcal{T}_x|f_y = 1, f'_x = 1\rangle = |f_y = 1, f'_x = -1\rangle,\\
&\mathcal{T}_y|f_y = 1, f'_x = 1\rangle = |f_y = -\mathrm{i}, f'_x = 1\rangle && \mathcal{T}_x\mathcal{T}_y|f_y = 1, f'_x = 1\rangle = 0,\\
&\mathcal{T}_y^2|f_y = 1, f'_x = 1\rangle = |f_y = -1, f'_x = 1\rangle && \mathcal{T}_x\mathcal{T}_y^2|f_y = 1, f'_x = 1\rangle = |f_y = -1, f'_x = -1\rangle,\\
&\mathcal{T}_y^3|f_y = 1, f'_x = 1\rangle = |f_y = i, f'_x = 1\rangle && \mathcal{T}_x\mathcal{T}_y^3|f_y = 1, f'_x = 1\rangle = 0.
\end{aligned} \tag{57}$$

There are exactly the transformation formulae of Oshikawa *et al* [10]. Note, however, that for our formula, there is a provision that after all the quasihole translation has been carried out, the two quasiholes are to be merged . As in equation (21), we have dropped the gauge transformation phase factor.

Oshikawa *et al* considered bases diagonalized with respect to either $\mathcal{T}_x$ or $\mathcal{T}_y$ [10]. These two bases are related by the modular S-matrix. In either basis, it can be shown that the transformation formulae of this subsection follow the Verlinde formula on diagonalization of fusion numbers by the S-matrix [15, 20].

3.2. Odd-spin-structure sector

We now need to consider the *odd*-spin structure in detail. On a torus, this is the case where the chiral Majorana fermion has periodic boundary conditions around both the generators. The action of a chiral Majorana fermion field ψ is

$$S = \frac{1}{2\pi}\int \mathrm{d}^2 z\, \psi \bar{\partial} \psi. \tag{58}$$

In performing the path integral, it is necessary to expand ψ in terms of the normalized c-number eigenmodes $\psi_{nm}(z, \bar{z})$ of $\bar{\partial}$. These must be doubly periodic, and so are

$$\psi_{nm}(z, \bar{z}) = \frac{1}{\sqrt{\operatorname{Im}\tau}} \exp \frac{\pi}{\operatorname{Im}\tau}[n(\tau\bar{z} - \bar{\tau}z) + m(z - \bar{z})]. \tag{59}$$

The resulting mode expansion is

$$\psi(z, \bar{z}) = \sum_{n,m\in\mathbb{Z}} a_{nm}\psi_{mn}(z, \bar{z}) = a_{00} + {\sum_{n,m\geqslant 0}}'[a_{nm}\psi_{nm}(z, \bar{z}) + a_{-n,-m}\psi_{-n,-m}(z, \bar{z})], \tag{60}$$

where a_n's are Grassman variables with $a_{-n,-m} \equiv a^*_{nm}$ and $\sum'_{nm}$ is a summation over non-negative integers that excludes the $n = m = 0$ term. It should be noted that, since ψ is holomorphic only as a result of the equation of motion, one cannot take it to be holomorphic in performing the path integral.

When the mode expansion equation (60) is inserted into the action of equation (58), we obtain

$$S = \frac{1}{2\pi} {\sum_{n,m\geqslant 0}}'(n\tau - m)a_{-n,-m}a_{nm}. \tag{61}$$

The Grassmann variable a_{00} does not appear in this action. However, a_{00} does appear in the integration measure

$$d[\psi] = \prod_{n,m\in\mathbb{Z}} \mathrm{d}a_{nm}. \tag{62}$$

This leads to the odd-spin-sector partition function vanishing [15, 16, 21]:

$$\int \mathrm{d}a_{00} \left({\prod_{n,m\geqslant 0}}' \mathrm{d}a_{nm}\, \mathrm{d}a_{-n,-m} \right) \exp(-S) = 0. \tag{63}$$

As in $\sum'_{nm}$ of equation (60), $\prod'_{nm}$ is a product of terms with non-negative integers n, m that excludes the $n = m = 0$ term. Now $\int a\, \mathrm{d}a = 1$ and $\int \mathrm{d}a = 0$ for any Grassmann variable a, and so the integration $\int F(a) \prod_{n,m} \mathrm{d}a_{nm}$ yields zero unless each a_{nm} appears exactly once in the integrand $F(a)$, and there is no a_{00} in the partition function integrand. For the same reason, any correlator of an even number of ψ's vanishes.

Correlators with *odd* number of ψ's can be nonzero. Consider the simplest case:

$$\langle\psi(z)\rangle \propto \int \mathrm{d}[\psi]\psi(z)\exp(-S) \propto \int a_{00}\,\mathrm{d}a_{00}\left(\prod_{n,m\geqslant 0}{}' \mathrm{d}a_{nm}\,\mathrm{d}a_{-n,-m}\right)\exp(-S)$$
$$= \int\left(\prod_{n,m}{}' \mathrm{d}a_{nm}\,\mathrm{d}a_{-n,-m}\right)\exp(-S) \neq 0. \tag{64}$$

Thus $\langle\psi(z)\rangle$ is a *nonzero* constant [15, 16]. The next simplest case is the three-point correlators, where three ψ fields take turns in occupying the zero mode:

$$\langle\psi(z_1)\psi(z_2)\psi(z_3)\rangle \propto \int \mathrm{d}[\psi]\psi(z_1)\psi(z_2)\psi(z_3)\exp(-S)$$
$$\propto \int\left(\prod_{n,m\geqslant 0}{}' \mathrm{d}a_{nm}\,\mathrm{d}a_{-n,-m}\right)\sum_{i,j\geqslant 0}{}' a_{ij}\psi_{ij}(z_1)\sum_{k,l\geqslant 0}{}' a_{kl}\psi_{kl}(z_2)\exp(-S)$$
$$+ \int\left(\prod_{n,m\geqslant 0}{}' \mathrm{d}a_{nm}\,\mathrm{d}a_{-n,-m}\right)\sum_{i,j\geqslant 0}{}' a_{ij}\psi_{ij}(z_2)\sum_{k,l\geqslant 0}{}' a_{kl}\psi_{kl}(z_3)\exp(-S)$$
$$+ \int\left(\prod_{n,m\geqslant 0}{}' \mathrm{d}a_{nm}\,\mathrm{d}a_{-n,-m}\right)\sum_{i,j\geqslant 0}{}' a_{ij}\psi_{ij}(z_3)\sum_{k,l\geqslant 0}{}' a_{kl}\psi_{kl}(z_1)\exp(-S)$$
$$= g_{++}(\mathbf{r}_1-\mathbf{r}_2) + g_{++}(\mathbf{r}_2-\mathbf{r}_3) + g_{++}(\mathbf{r}_3-\mathbf{r}_1), \tag{65}$$

where [7, 8]

$$g_{++}(\mathbf{r}) = \frac{\vartheta_1'(z)}{\vartheta_1(z)} + \frac{2\pi\mathrm{i}y}{L_y} \tag{66}$$

is a modified Green function satisfying

$$\bar{\partial}g_{++}(\mathbf{r}) = \pi\delta^{(2)}(\mathbf{r}) - \frac{\pi}{\operatorname{Im}\tau}. \tag{67}$$

This Green function contains $y = (z+\bar{z})/2$ and is not holomorphic. $\bar{z}$'s, however, cancel in sum of *three* Green functions appearing in equation (65). The three-point correlator is, therefore, holomorphic.

In the odd-spin structure, and for N_e odd, the general N_e-point chiral Majorana fermion correlator is

$$\left\langle\psi(z_1)\ldots\psi\left(z_{N_e}\right)\right\rangle \propto \Psi_{++}\left(z_1,\ldots,z_{N_e}\right)$$
$$= \frac{1}{2^{(N_e-1)/2}[(N_e-1)/2]!}\sum_{P\in S_{N_e}}\mathrm{sgn}(P)\prod_{i=1}^{(N_e-1)/2} g_{++}(\mathbf{r}_{P(2i-1)}-\mathbf{r}_{P(2i)}). \tag{68}$$

Here P runs over all permutations of N_e objects. Again, at first sight, this equation looks neither chiral nor holomorphic. However, Read and his coworkers [7, 8] asserted that the cancellation in equation (65) generalizes to

$$\Psi_{++}\left(z_1,\ldots,z_{N_e}\right) = \frac{1}{2^{(N_e-1)/2}[(N_e-1)/2]!}\sum_{P\in S_{N_e}}\mathrm{sgn}(P)\prod_{i=1}^{(N_e-1)/2}\frac{\vartheta_1'\left(z_{P(2i-1)}-z_{P(2i)}\right)}{\vartheta_1\left(z_{P(2i-1)}-z_{P(2i)}\right)}. \tag{69}$$

This rather non-obvious cancellation of $\bar{z}$'s will be proved in the appendices.

Equation (68) was first constructed by Read and Green in their work on spinless $p + \mathrm{i}p$ superconductor [8]. They have pointed out that the existence of the $\mathbf{k} = 0$ mode, which is an almost exact analogue of the zero mode discussed here, requires the number of electrons to be odd, as long as it is energetically favourable to have this $\mathbf{k} = 0$ mode occupied.

We emphasize again that these periodic and antiperiodic 'boundary conditions' for $\psi(z)$'s are *not* the boundary conditions of the physical electrons. The physical boundary conditions are those of equations (1) and (3). The boundary conditions for $\psi(z)$'s affect only the manner of BCS pairing. The best way of regarding equation (68) is that this is the correct replacement for the Pfaffian when the number of electrons is odd. The centre-of-mass wavefunctions ensure that all Moore–Read states on a torus have the same physical boundary conditions under translation of electrons around generators, regardless of whether the number of electrons is even or odd. This means that $\tilde{F}^{(m)}_{\mathrm{cm}}$ of equation (28) is the centre-of-mass wavefunction when the number of electrons on the torus is odd. The holomorphic part of the ground-state wavefunction in this case is therefore

$$f^{(m)}_{\mathrm{odd}}(z_1, \ldots, z_{N_e}) = \tilde{F}^{(m)}_{\mathrm{cm}}\left(\sum_i z_i\right)\Psi_{++}(z_1, \ldots, z_{N_e})\prod_{i<j}[\vartheta_1(z_i - z_j)]^2. \tag{70}$$

Obtaining the wavefunctions with one charge $e/2$ quasihole with one quantum flux is now straightforward. It is in essentially the same form as equation (31):

$$f^{(m)}_{\mathrm{odd}}(z_1, \ldots, z_{N_e}; w) = \tilde{F}^{(m)}_{\mathrm{cm}}\left(\sum_i z_i + \frac{w}{2}\right)\Psi_{++}(z_1, \ldots, z_{N_e}) \times \prod_i \vartheta_1(z_i - w)\prod_{i<j}[\vartheta_1(z_i - z_j)]^2. \tag{71}$$

How one do we fractionalize the quasihole of equation (71) to obtain two charge $e/4$ quasiholes? The first formula of equation (39), being independent of the parity of number of ψ's, still holds. Also note if we define $\tilde{M}^{\mathrm{odd}}_{ij} = -\tilde{M}^{\alpha=(1/2,1/2)}_{ij}$ then

$$\lim_{z_i\to z_j}\frac{(z_i - z_j)\vartheta_1'(0)\tilde{M}^{\mathrm{odd}}_{ij}}{[\vartheta_1(z_i - w_1)\vartheta_1(z_i - w_2)\vartheta_1(z_j - w_1)\vartheta_1(z_j - w_2)]^{1/2}} = 1. \tag{72}$$

Therefore, in the odd-spin structure for N_e odd, the chiral Ising field correlator of equation (40) becomes

$$\begin{aligned}\left\langle\psi(z_1)\cdots\psi(z_{N_e})\sigma(w_1)\sigma(w_2)\right\rangle_{\alpha=(1/2,1/2)} &= \prod_{i=1}^{N_e}[\vartheta_1(z_i - w_1)\vartheta_1(z_i - w_2)]^{-1/2}\\ &\times\sum_{P\in S_{N_e}}\mathrm{sgn}(P)\left[\vartheta_1\left(z_{P(N_e)} - w_1\right)\vartheta_1\left(z_{P(N_e)} - w_2\right)\right]^{1/2}\\ &\times\left\langle\psi\left(z_{P(N_e)}\right)\sigma(w_1)\sigma(w_2)\right\rangle_{\alpha=(1/2,1/2)}\\ &\times\prod_{j=1}^{(N_e-1)/2}\left[\vartheta_1'(0)\tilde{M}^{\mathrm{odd}}_{P(2j-1),P(2j)}\right].\end{aligned} \tag{73}$$

Note that one now needs to calculate $\langle\psi\sigma\sigma\rangle_{\alpha=(1/2,1/2)}$ rather than $\langle\sigma\sigma\rangle_{\alpha=(1/2,1/2)}$. This leads to the conclusion that with two $e/4$ quasiholes at w_1 and w_2, equation (71) should be modified into

$$\tilde{\Psi}_{++}\big(z_1, \ldots, z_{N_e}; w_1, w_2\big) = \text{const} \sum_{P \in S_{N_e}} \text{sgn}(P)[\vartheta_1(w_1 - w_2)]^{1/8} \times \big[\vartheta_1\big(z_{P(N_e)} - w_1\big)\big]^{1/2}\big[\vartheta_1\big(z_{P(N_e)} - w_2\big)\big]^{1/2} \times \big\langle\psi\big(z_{P(N_e)}\big)\sigma(w_1)\sigma(w_2)\big\rangle_{\alpha=(1/2,1/2)} \prod_{i=1}^{(N_e-1)/2} \big[\tilde{M}^{\text{odd}}_{P(2i-1),P(2i)}\big]. \tag{74}$$

We compute the necessary Ising correlator in the appendices. We find that

$$\langle\psi(z)\sigma(w_1)\sigma(w_2)\rangle_{\alpha=(1/2,1/2)} \propto \left[\frac{1}{\vartheta_1(w_{12})}\right]^{1/8} \times \left[\vartheta_1'(w_{12}/2) + \frac{1}{2}\vartheta_1(w_{12}/2)\left(\frac{\vartheta_1'(z-w_1)}{\vartheta_1(z-w_1)} - \frac{\vartheta_1'(z-w_2)}{\vartheta_1(z-w_2)}\right)\right]^{1/2}. \tag{75}$$

Consequently, ignoring a multiplicative constant, we have

$$\tilde{\Psi}_{++}\big(z_1, \ldots, z_{N_e}; w_1, w_2\big) = \sum_{P \in S_{N_e}} \text{sgn}(P)\big[h\big(z_{P(N_e)}; w_1, w_2\big)\big]^{1/2} \prod_{i=1}^{(N_e-1)/2} \big[\tilde{M}^{\text{odd}}_{P(2i-1),P(2i)}\big], \tag{76}$$

where

$$h(z; w_1, w_2) = \vartheta_1'(w_{12}/2)\vartheta_1(z-w_1)\vartheta_1(z-w_2) + \tfrac{1}{2}\vartheta_1(w_{12}/2)(\vartheta_1'(z-w_1)\vartheta_1(z-w_2) - \vartheta_1'(z-w_2)\vartheta_1(z-w_1)). \tag{77}$$

The holomorphic part of the wavefunction with two charge $e/4$ quasiholes is

$$\begin{aligned} f^{(m)}_{\text{odd}}\big(z_1, \ldots, z_{N_e}; w_1, w_2\big) &= \tilde{F}^{(m)}_{\text{cm}}\left(\sum_i z_i + \frac{w_1+w_2}{4}\right)\tilde{\Psi}_{++}\big(z_1, \ldots, z_{N_e}; w_1, w_2\big)\prod_{i<j}[\vartheta_1(z_i - z_j)]^2 \\ &= \tilde{F}^{(m)}_{\text{cm}}\left(\sum_i z_i + \frac{w_1+w_2}{4}\right)\prod_{i<j}[\vartheta_1(z_i - z_j)]^2 \\ &\quad\times \sum_{P \in S_{N_e}} \text{sgn}(P)\big[h\big(z_{P(N_e)}; w_1, w_2\big)\big]^{1/2} \prod_{k=1}^{(N_e-1)/2} \big[\tilde{M}^{\text{odd}}_{P(2k-1),P(2k)}\big]. \end{aligned} \tag{78}$$

It is not obvious that this wavefunction equation (78) is analytic in electron coordinates. We need to show that $[h(z; w_1, w_2)]^{1/2}$ is an analytic function of z. The proof of the analyticity of $[h(z; w_1, w_2)]^{1/2}$ can be presented in two steps. First, note that from

$$\begin{aligned} h(z+1; w_1, w_2) &= h(z; w_1, w_2) \\ h(z+\tau; w_1, w_2) &= \exp[-2\pi \mathrm{i}(2z - w_1 - w_2 + \tau)]h(z; w_1, w_2) \end{aligned} \tag{79}$$

z of $h(z; w_1, w_2)$ should have two zeros in the first principal region. Second, note that both $h(z; w_1, w_2)$ and $\partial_z h(z; w_1, w_2)$ vanish at $z = (w_1 + w_2)/2$. This shows that there is actually a *double* zero at $z = (w_1 + w_2)/2$ and this is enough to ensure that $[h(z; w_1, w_2)]^{1/2}$ is indeed analytic in z.

We should also verify that the odd-spin-sector wavefunction with two $e/4$ quasiholes goes smoothly into the corresponding wavefunction with one $e/2$ quasihole (up to a overall multiplicative constant) as the $e/4$ quasiholes merge. In other words, in the limit that

$w_1, w_2 \to w$, does $\tilde{\Psi}_{++}(z_1, \ldots, z_{N_e}; w_1, w_2)$ of equation (76) become proportional to $g_{++}(z_1, \ldots, z_{N_e}) \prod_i \vartheta_1(z_i - w)$? From equation (50),

$$\lim_{w_1, w_2 \to w} \tilde{\Psi}_{++}(z_1, \ldots, z_{N_e}; w_1, w_2) = [\vartheta_1'(0)]^{1/2} \prod_i \vartheta_1(z_i - w) \sum_{P \in S_{N_e}} \mathrm{sgn}(P) \prod_{i=1}^{(N_e-1)/2} \frac{1}{\vartheta_1'(0)}$$
$$\times \left[\frac{\vartheta_1'(z_{P(2i-1)} - z_{P(2i)})}{\vartheta_1(z_{P(2i-1)} - z_{P(2i)})} - \left(\frac{\vartheta_1'(z_{P(2i-1)} - w)}{\vartheta_1(z_{P(2i-1)} - w)} - \frac{\vartheta_1'(z_{P(2i)} - w)}{\vartheta_1(z_{P(2i)} - w)} \right) \right]$$
$$= \mathrm{const} \prod_i \vartheta_1(z_i - w) \sum_{P \in S_{N_e}} \mathrm{sgn}(P) \prod_{i=1}^{(N_e-1)/2} \frac{\vartheta_1'(z_{P(2i-1)} - z_{P(2i)})}{\vartheta_1(z_{P(2i-1)} - z_{P(2i)})}$$
$$= \mathrm{const}\Psi_{++}(z_1, \ldots, z_{N_e}) \prod_i \vartheta_1(z_i - w). \tag{80}$$

The cancellation of the w-dependent terms other than $\prod_i \vartheta_1(z_i - w)$ occurs for exactly same reason as the one in equation (69). This cancellation will be proved in the appendices.

From

$$\begin{aligned} h(z; w_1 \pm 1, w_2) &= \pm\vartheta_2'(w_{12}/2)\vartheta_1(z - w_1)\vartheta_1(z - w_2) \\ &\quad \pm \tfrac{1}{2}\vartheta_2(w_{12}/2)(\vartheta_1'(z - w_1)\vartheta_1(z - w_2) - \vartheta_1'(z - w_2)\vartheta_1(z - w_1)), \\ h(z; w_1 \pm \tau, w_2) &= \pm\, \mathrm{i}\mathrm{e}^{-\mathrm{i}\pi\tau/4}\, \mathrm{e}^{\mp \mathrm{i}\pi w_{12}/2}\, \mathrm{e}^{-\mathrm{i}\pi\tau}\, \mathrm{e}^{\pm 2\pi \mathrm{i} z}[\vartheta_4'(w_{12}/2)\vartheta_1(z - w_1)\vartheta_1(z - w_2) \\ &\quad + \tfrac{1}{2}\vartheta_4(w_{12}/2)(\vartheta_1'(z - w_1)\vartheta_1(z - w_2) - \vartheta_1'(z - w_2)\vartheta_1(z - w_1))], \end{aligned} \tag{81}$$

it is clear that after one quasihole is translated around a generator the wavefunction vanishes when two quasiholes are brought together.

As in the even-spin-structure case, this vanishing can be attributed to the change in the fusion channel of two σ's. This is so since

$$\lim_{w_1 \to w_2} \langle \psi(z)\sigma(w_1)\sigma(w_2)\rangle_{\alpha=(1/2,1/2)} \sim w_{12}^{-1/8}, \tag{82}$$

but after the translation $w_1 \to w_1 \pm 1$ or $w_1 \to w_1 \pm \tau$, equation (81) tells us that the same correlator vanishes as $w_{12}^{3/8}$ when $w_1 \to w_2$. From the chiral Ising operator product expansion, equation (54), one can see two σ's fuse to $\mathbb{I}$ in $\langle \psi\sigma\sigma\rangle_{\alpha=(1/2,1/2)}$, but after the $w_1 \to w_1 \pm 1$ or $w_1 \to w_1 \pm \tau$ translation they fuse to ψ. The argument made in the last subsection that this change in the fusion channel is accompanied by the change in the parity of the number of the electrons in the ground state remains valid. Since the number of electrons in the system does not change, a quasiparticle excitation must have been created. Note that the different monodromy results of the odd-spin structure arise from the fact that here $\langle \psi\sigma\sigma\rangle$ plays the role of $\langle \sigma\sigma\rangle$ in the even-spin structures. We see that the different monodromy outcomes originate from the different parities of the electron number.

From the monodromy outcomes we have obtained, the ground states of the odd-spin structure are eigenstates of both $\mathcal{T}_x$ and $\mathcal{T}_y$. However, in the even-spin structures, no ground state is simultaneously an eigenstate of both $\mathcal{T}_x$ and $\mathcal{T}_y$. The difference comes from the fact that, in the language of the critical Ising model, the total topological charge of the system is ψ in the odd-spin structure whereas it is $\mathbb{I}$ in the even-spin structures. That means that the modular S-matrix of the system for the odd-spin structure differs from that of the even-spin structures—S^{ψ} for the former and $S^{\mathbb{I}}$ for the latter [22, 23].

This discussion on total topological charge indicates that the odd-spin-structure sector exists only because electrons that make up a quantum Hall system are fermions. On the other hand, a system consisting of bosons, as in the string net of Levin and Wen [24], one cannot have a ground state with total topological charge of ψ. In such a system, one would still have 'forbidden transition' due to change in the fusion channel, but this change is not tied with the change in the parity of number of particles in the ground state in the manner discussed here. The change in the fusion channel can be attributed to the change in the parity of number of particles in the ground state only when the particles are fermions and the ground state has BCS pairing.

4. Conclusion and discussion

In this paper, we constructed the two-quasihole wavefunctions for the Moore–Read quantum Hall state on a torus for both even and odd-spin structures. Conformal field theory calculations enable us to obtain the complete dependence of these wavefunctions on quasihole coordinates. We showed that the number of electrons in a ground state must be odd in the odd-spin structure and even in the even-spin structures. By noting that the boundary conditions of each electron remain that of equation (1) for all topologically degenerate ground states, we obtained the explicit expressions for the centre-of-mass wavefunctions in all cases. Analytic continuation allowed us to obtain the monodromy matrix describing the effect on the space degenerate ground states of taking quasiholes around the torus generators. The effects are in agreement with those obtained by Oshikawa *et al*. In this process, we demonstrated that otherwise anticipated transitions are forbidden because they involve a change in the fusion channel of the quasiholes. This reflects a change in the parity of the number of electrons that can reside in the ground state. Since the number of electrons is conserved, the operations that might have resulted in these forbidden operations take us out of the space of degenerate ground states and into the space of excited states.

Several extensions of our analysis are possible. Theta functions can be generalized to higher genus Riemann surface [25], and these functions will naturally be ingredients of wavefunctions in such topology. Such wavefunctions for Laughlin states had been studied [26]. With these wavefunctions are found for the Moore–Read state, it should be possible to extend our calculations to the higher genus Riemann surface considered by Oshikawa *et al* [10]. It would also be valuable to extend this analysis to consider the next simplest non-Abelian quantum Hall states, the Read–Rezayi parafermion states. The number of degenerate ground states has already been found [27]. The challenge is to work out the wavefunction on the compact Riemann surface using the conformal field theory analysis, thus extending the recent wavefunction construction by Ardonne and Schoutens on the plane [28].

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Appendix A. Translation eigenvalues of wavefunctions

Following equation (55), $\mathcal{T}_x^{-2}$ and $\mathcal{T}_y^{4}$ can be regarded as implementing $\mathbf{R}_1 \to \mathbf{R}_1 - L_x\hat{\mathbf{x}}$ twice and $\mathbf{R}_1 \to \mathbf{R}_1 + L_y\hat{\mathbf{y}}$ four times, respectively, on two-quasihole wavefunctions $\Psi^{(a,m)}(\mathbf{r}_1,\ldots,\mathbf{r}_{N_e};\mathbf{R}_1,\mathbf{R}_2)$. For $a=3$, repeated application of equation (45) leads to

$$\begin{aligned}
&\mathcal{T}_y^{4}:\Psi^{(a=2,m)}(\mathbf{r}_1,\ldots,\mathbf{r}_{N_e};\mathbf{R}_1,\mathbf{R}_2)\to\exp(-\mathrm{i}L_y\xi_1/l^2)\Psi^{(a=2,m)}(\mathbf{r}_1,\ldots,\mathbf{r}_{N_e};\mathbf{R}_1,\mathbf{R}_2),\\
&\mathcal{T}_x^{-2}:\Psi^{(a=3,m)}(\mathbf{r}_1,\ldots,\mathbf{r}_{N_e};\mathbf{R}_1,\mathbf{R}_2)\to \mathrm{e}^{\mathrm{i}\pi m}\Psi^{(a=3,m)}(\mathbf{r}_1,\ldots,\mathbf{r}_{N_e};\mathbf{R}_1,\mathbf{R}_2),\\
&\mathcal{T}_y^{4}:\Psi^{(a=3,m)}(\mathbf{r}_1,\ldots,\mathbf{r}_{N_e};\mathbf{R}_1,\mathbf{R}_2)\to\exp(-\mathrm{i}L_y\xi_1/l^2)\Psi^{(a=3,m)}(\mathbf{r}_1,\ldots,\mathbf{r}_{N_e};\mathbf{R}_1,\mathbf{R}_2),\\
&\mathcal{T}_x^{-2}:\Psi^{(a=4,m)}(\mathbf{r}_1,\ldots,\mathbf{r}_{N_e};\mathbf{R}_1,\mathbf{R}_2)\to \mathrm{e}^{\mathrm{i}\pi m}\Psi^{(a=4,m)}(\mathbf{r}_1,\ldots,\mathbf{r}_{N_e};\mathbf{R}_1,\mathbf{R}_2).
\end{aligned}\tag{A.1}$$

To apply these operations on $a=2$ and $a=4$ states, one also needs to consider the transformation of $\tilde{f}^{(m)}(z_1,\ldots,z_{N_e};w_1,w_2)$ of equation (49), even if this does not qualify as a Moore–Read state wavefunction. From the standard theta function identities

$$\begin{aligned}
&\vartheta_1(z-1/2)=-\vartheta_2(z),\\
&\vartheta_1(z+\tau/2)=\mathrm{i}\exp[-\mathrm{i}\pi(z+\tau/4)]\vartheta_4(z),
\end{aligned}\tag{A.2}$$

and the centre-of-mass wavefunction transformation

$$\begin{aligned}
&\tilde{F}^{(m)}(z-1/4)=F_{\mathrm{cm}}^{(a=2,m)}(z),\\
&\tilde{F}^{(m)}(z+\tau/4)=\mathrm{e}^{\mathrm{i}\pi(N_s-2)/4}\exp[-\mathrm{i}\pi(z+\tau/8)]F_{\mathrm{cm}}^{(a=4,m)}(z),
\end{aligned}\tag{A.3}$$

one obtains

$$\begin{aligned}
&w_1\to w_1-1:\tilde{f}^{(m)}(z_1,\ldots,z_{N_e};w_1,w_2)\to(-1)^{N_e/2}f^{(a=2,m)}(z_1,\ldots,z_{N_e};w_1,w_2),\\
&w_1\to w_1+\tau:\tilde{f}^{(m)}(z_1,\ldots,z_{N_e};w_1,w_2)\to \mathrm{e}^{-\mathrm{i}\pi/4}\exp[-\mathrm{i}\pi N_s(w_1/2+\tau/4)]\\
&\qquad\times f^{(a=4,m)}(z_1,\ldots,z_{N_e};w_1,w_2).
\end{aligned}\tag{A.4}$$

Repeated application of equations (45) and (A.4) leads to

$$\begin{aligned}
&\mathcal{T}_x^{-2}:\Psi^{(a=2,m)}(\mathbf{r}_1,\ldots,\mathbf{r}_{N_e};\mathbf{R}_1,\mathbf{R}_2)\to -\mathrm{i}\,\mathrm{e}^{\mathrm{i}\pi(m-1)}\Psi^{(a=2,m)}(\mathbf{r}_1,\ldots,\mathbf{r}_{N_e};\mathbf{R}_1,\mathbf{R}_2),\\
&\mathcal{T}_y^{4}:\Psi^{(a=4,m)}(\mathbf{r}_1,\ldots,\mathbf{r}_{N_e};\mathbf{R}_1,\mathbf{R}_2)\to -\exp(-\mathrm{i}L_y\xi_1/l^2)\Psi^{(a=4,m)}(\mathbf{r}_1,\ldots,\mathbf{r}_{N_e};\mathbf{R}_1,\mathbf{R}_2).
\end{aligned}\tag{A.5}$$

Appendix B. Proof for cancellation in summation

Consider a function $G(\mathbf{r},\mathbf{r}')$ odd under exchange of $\mathbf{r}$ and $\mathbf{r}'$. Suppose this function can be expressed as a sum of two function $g_1(\mathbf{r},\mathbf{r}')$ and $g_2(\mathbf{r},\mathbf{r}')$; $g_2(\mathbf{r},\mathbf{r}')$ has a further property that $g_2(\mathbf{r}_1,\mathbf{r}_2)+g_2(\mathbf{r}_2,\mathbf{r}_3)+g_2(\mathbf{r}_3,\mathbf{r}_1)=0$. Certain functions in section 3, such as $g_{++}(\mathbf{r})$ in equation (68) and $\tilde{M}_{ij}^{\mathrm{odd}}$ in the limit $w_1,w_2\to w$ as in equation (80) (when regarded as a function of two electron coordinates), are of this form. In order that the assertions made in equations (68) and (80) hold, it is necessary to show that for N odd

$$\sum_{P\in S_N}\mathrm{sgn}(P)\prod_{i=1}^{(N-1)/2}G(\mathbf{r}_{P(2i-1)},\mathbf{r}_{P(2)})=\sum_{P\in S_N}\mathrm{sgn}(P)\prod_{i=1}^{(N-1)/2}g_1(\mathbf{r}_{P(2i-1)},\mathbf{r}_{P(2i)}).\tag{B.1}$$

One can easily see that it should be so for $N=3$ from $G(\mathbf{r}_1,\mathbf{r}_2)+G(\mathbf{r}_2,\mathbf{r}_3)+G(\mathbf{r}_3,\mathbf{r}_1)=g_1(\mathbf{r}_1,\mathbf{r}_2)+g_1(\mathbf{r}_2,\mathbf{r}_3)+g_1(\mathbf{r}_3,\mathbf{r}_1)$, due to the property of g_2 mentioned above. Suppose it holds for $N=2k+1$ where k is some positive integer. If it can be shown from this assumption that equation (B.1) holds for $N=2k+3$, then we have a proof by induction.

To apply induction, a new permutation $P' \in S_{2k+1}$ needs be introduced:

$$\sum_{P\in S_{2k+3}} \mathrm{sgn}(P)\prod_{i=1}^{k+1} G(\mathbf{r}_{P(2i-1)},\mathbf{r}_{P(2i)}) = 2\sum_{m<n}(-1)^{m-n-1}G(\mathbf{r}_m,\mathbf{r}_n) \times \sum_{P'\in S_{2k+1}} \mathrm{sgn}(P')\prod_{i=1}^{k} G(\mathbf{r}_{P'(2i-1)},\mathbf{r}_{P'(2i)}), \tag{B.2}$$

where m, n are integers between 1 and $2k+3$. This P' is a permutation of integers between 1 and $2k+3$ *except m and n*. Since the assumption had been made that equation (B.1) holds for $N = 2k+1$, equation (B.2) means

$$\sum_{P\in S_{2k+3}} \mathrm{sgn}(P)\prod_{i=1}^{k+1} G(\mathbf{r}_{P(2i-1)},\mathbf{r}_{P(2i)}) = 2\sum_{m<n}(-1)^{m-n-1}G(\mathbf{r}_m,\mathbf{r}_n) \times \sum_{P'\in S_{2k+1}} \mathrm{sgn}(P')\prod_{i=1}^{k} g_1(\mathbf{r}_{P'(2i-1)},\mathbf{r}_{P'(2i)}). \tag{B.3}$$

Comparing equation (B.2) and equation (B.3) shows that equation (B.3) means we can replace all but one G into g_1. However, this leads to the conclusion that the last remaining G can also be replaced by g_1:

$$\begin{aligned}
&\sum_{P\in S_{2k+3}} \mathrm{sgn}(P)\prod_{i=1}^{k+1} G(\mathbf{r}_{P(2i-1)},\mathbf{r}_{P(2i)}) \\
&\quad= \sum_{P\in S_{2k+3}} \mathrm{sgn}(P)G(\mathbf{r}_{P(2k+1)},\mathbf{r}_{P(2k+2)})\prod_{i=2}^{k+1} g_1(\mathbf{r}_{P(2i-1)},\mathbf{r}_{P(2i)}) \\
&\quad= \frac{1}{3}\sum_{P\in S_{2k+3}} \mathrm{sgn}(P)[G(\mathbf{r}_{P(2k+1)},\mathbf{r}_{P(2k+2)}) + G(\mathbf{r}_{P(2k+2)},\mathbf{r}_{P(2k+3)}) \\
&\qquad + G(\mathbf{r}_{P(2k+3)},\mathbf{r}_{P(2k+1)})]\prod_{i=2}^{k+1} g_1(\mathbf{r}_{P(2i-1)},\mathbf{r}_{P(2i)}) \\
&\quad= \frac{1}{3}\sum_{P\in S_{2k+3}} \mathrm{sgn}(P)[g_1(\mathbf{r}_{P(2k+1)},\mathbf{r}_{P(2k+2)}) + g_1(\mathbf{r}_{P(2k+2)},\mathbf{r}_{P(2k+3)}) \\
&\qquad + g_1(\mathbf{r}_{P(2k+3)},\mathbf{r}_{P(2k+1)})]\prod_{i=2}^{k+1} g_1(\mathbf{r}_{P(2i-1)},\mathbf{r}_{P(2i)}) \\
&\quad= \sum_{P\in S_{2k+3}} \mathrm{sgn}(P)\prod_{i=1}^{k+1} g_1(\mathbf{r}_{P(2i-1)},\mathbf{r}_{P(2i)}).
\end{aligned} \tag{B.4}$$

Appendix C. Critical Ising correlators on torus

To calculate

$$\langle\psi(z)\sigma(w_1)\sigma(w_2)\rangle_{\alpha=(1/2,1/2)}, \tag{C.1}$$

first note the connection between the correlator in the chiral Ising theory and the full Ising theory:

$$|\langle\psi(z)\sigma(w_1)\sigma(w_2)\rangle|^2 = \langle\varepsilon(z,\bar{z})\sigma(w_1,\bar{w}_1)\sigma(w_2,\bar{w}_2)\rangle. \tag{C.2}$$

So $\langle\psi\sigma\sigma\rangle$ can be computed by taking the holomorphic part of $\langle\varepsilon\sigma\sigma\rangle$. By the Ising model bosonization formula [15, 30]

$$\langle\varepsilon(z,\bar z)\sigma(w_1,\bar w_1)\sigma(w_2,\bar w_2)\rangle^2 = -2\left\langle \partial\phi(z)\bar\partial\phi(\bar z)\cos\frac{\phi(w_1,\bar w_1)}{2}\cos\frac{\phi(w_2,\bar w_2)}{2}\right\rangle$$
$$= -\frac{1}{2}[\langle\partial\phi(z)\bar\partial\phi(\bar z)\exp(\mathrm{i}\phi(w_1,\bar w_1)/2)\exp(-\mathrm{i}\phi(w_2,\bar w_2)/2)\rangle$$
$$+\langle\partial\phi(z)\bar\partial\phi(\bar z)\exp(-\mathrm{i}\phi(w_1,\bar w_1)/2)\exp(\mathrm{i}\phi(w_2,\bar w_2)/2)\rangle], \tag{C.3}$$

where $\phi(z,\bar z)$ is a free boson field. It should be emphasized that there is an intricacy hidden behind equation (C.3). Critical Ising fields on torus can be bosonized only into a *compactified* free boson. A boson with compactification radius $r = 1$ should have boundary condition

$$\phi^{(m,m')}(z+1,\bar z+1) = \phi^{m,m'}(z,\bar z) + 2\pi m, \qquad \phi^{(m,m')}(z+\tau,\bar z+\bar\tau) = \phi^{m,m'}(z,\bar z) + 2\pi m', \tag{C.4}$$

where m, m' are integers. This winding comes entirely from the zero mode, so one can express the free boson field ϕ as a sum of this zero mode and 'free part' $\hat\phi$, that is, nonzero modes:

$$\phi^{(m,m')}(z,\bar z) = \frac{\pi}{\mathrm{i}\tau_2}[m(\tau\bar z - \bar\tau z) + m'(z-\bar z)] + \hat\phi(z,\bar z), \tag{C.5}$$

where $\tau = \tau_1 + \mathrm{i}\tau_2$ with τ_1, τ_2 being real. (Note that in this appendix, the first principal region is a parallelogram and not necessarily a square and nonzero τ_1 is considered.) This decomposition of the boson field leads to the following decomposition of the action [15, 21]:

$$S[\phi] = (1/8\pi)\int(\partial\phi)(\bar\partial\phi)$$
$$= S[\phi_0] + S[\hat\phi] - (1/4\pi)\int\hat\phi\Delta\phi_0 = S[\phi_0] + S[\hat\phi], \tag{C.6}$$

where ϕ_0 refers to the zero mode terms of equation (C.5). This result is due to the vanishing of the Laplacian of ϕ_0. This also means that the boson partition function Z^{bos} factorizes into the zero mode part Z_0 and the 'free part' $\hat Z$: $Z^{\mathrm{bos}} = Z_0\hat Z$. Since in calculating Z_0, all m, m' need to be summed over, one obtains [15, 16, 21]

$$Z_0 = \sum_{m,m'} Z_0^{(m,m')}, \tag{C.7}$$

where $Z_0^{(m,m')} = \exp[-(\pi|m\tau - m'|^2/2\tau_2)]$.

To compute a full boson correlator, one should calculate the correlator first for the winding sector m, m' and then sum over all possible winding numbers. However, this does not suffice for the purpose here, which is to calculate $\langle\psi\sigma\sigma\rangle_{\alpha=(1/2,1/2)}$. The question is how to extract out the portion of the correlator that corresponds to $\alpha = (1/2, 1/2)$ sector of the Ising model.

The answer can be obtained from converting expressing theta functions in terms of $Z_0^{(m,m')}$ of equation (C.7). Following Di Francesco *et al* [16]

$$\left|\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(z|\tau)\right|^2 = \sum_{n,\bar n}\exp(\mathrm{i}\pi[\tau(n+1/2)^2 - \bar\tau(\bar n+1/2)^2])\,\mathrm{e}^{2\pi\mathrm{i}[(n+1/2)(z+1/2)-(\bar n+1/2)(\bar z+1/2)]}$$
$$= \left(\sum_{m\in2\mathbb{Z}+1}\sum_{q\in2\mathbb{Z}} + \sum_{m\in2\mathbb{Z}}\sum_{q\in2\mathbb{Z}+1}\right)(-1)^m$$
$$\times\exp(\mathrm{i}\pi[mq\tau_1 + (i/2)(m^2+q^2)\tau_2])\,\mathrm{e}^{\pi\mathrm{i}[m(z+\bar z)+q(z-\bar z)]}, \tag{C.8}$$

where $m = n - \bar{n}$ and $q = n + \bar{n} + 1$. Applying the *Poisson resummation formula*

$$\sum_n \exp(-\pi a n^2 + bn) = \frac{1}{\sqrt{a}} \sum_k \exp\left[-\frac{\pi}{a}(k + b/2\pi \mathrm{i})^2\right] \tag{C.9}$$

for the summation over q in equation (C.8) leads to

$$\begin{aligned}
|\vartheta_1(z)|^2 = &\frac{-1}{\sqrt{2\tau_2}} \sum_{m\in 2\mathbb{Z}+1} \mathrm{e}^{-\pi m^2\tau_2/2}\, \mathrm{e}^{\pi \mathrm{i} m(z+\bar{z})} \left(\sum_{m'} \exp[-(\pi/2\tau_2)(m' - m\tau_1 - z + \bar{z})^2]\right) \\
&+ \frac{\mathrm{e}^{\pi \mathrm{i}(z-\bar{z}+\mathrm{i}\tau_2/2)}}{\sqrt{2\tau_2}} \sum_{m\in 2\mathbb{Z}} \mathrm{e}^{-\pi m^2\tau_2/2}\, \mathrm{e}^{\pi \mathrm{i} m(z+\bar{z}+\tau_1)} \\
&\times \left(\sum_{m'} \exp[-(\pi/2\tau_2)(m' - m\tau_1 - z + \bar{z} - \mathrm{i}\tau_2)^2]\right) \\
= &-\frac{\exp[-\pi(z-\bar{z})^2/2\tau_2]}{\sqrt{2\tau_2}} \sum_{m,m'} (-1)^{(m+1)(m'+1)} Z_0^{(m,m')} \exp[\mathrm{i}\phi_0^{(m,m')}(z,\bar{z})],
\end{aligned} \tag{C.10}$$

where $\phi_0^{(m,m')}(z,\bar{z}) = (\pi/\mathrm{i}\tau_2)[m(\tau\bar{z} - \bar{\tau}z) + m'(z - \bar{z})]$ as in equation (C.5). (Note that the definition of the Jacobi theta functions, equation (9) is also used.)

Equations (C.8) and (C.10) tell us that when summing over different winding numbers m and m', weighting different winding sector by the sign factor $-(-1)^{(m+1)(m'+1)}$ would result in extracting out the $\alpha = (1/2, 1/2)$ sector of the Ising model. It is now possible to calculate the boson correlators of equation (C.3), e.g.,

$$\begin{aligned}
&\langle \partial\phi(z)\bar{\partial}\phi(\bar{z}) \exp(\mathrm{i}\phi(w_1,\bar{w}_1)/2) \exp(-\mathrm{i}\phi(w_2,\bar{w}_2)/2)\rangle_{\alpha=(1/2,1/2)} \\
&\quad= \frac{-1}{Z_0\hat{Z}} \int \mathcal{D}\hat{\phi}\, \mathrm{e}^{-S[\hat{\phi}]} \sum_{m,m'} (-1)^{(m+1)(m'+1)} Z_0^{(m,m')} \partial\phi^{(m,m')}(z)\bar{\partial}\phi^{(m,m')}(\bar{z}) \\
&\qquad\times \exp(\mathrm{i}\phi^{(m,m')}(w_1,\bar{w}_1)/2) \exp(-\mathrm{i}\phi^{(m,m')}(w_2,\bar{w}_2)/2) \\
&\quad= \frac{-1}{Z_0} \sum_{m,m'} (-1)^{(m+1)(m'+1)} Z_0^{(m,m')} \exp\left[\mathrm{i}\phi_0^{(m,m')}(w_{12}/2, \bar{w}_{12}/2)\right] \\
&\qquad\times \Big[\partial\phi_0^{(m,m')}(z)\bar{\partial}\phi_0^{(m,m')}(\bar{z}) \langle \exp(\mathrm{i}\hat{\phi}(w_1,\bar{w}_1)/2) \exp(-\mathrm{i}\hat{\phi}(w_2,\bar{w}_2)/2)\rangle \\
&\qquad+ \partial\phi_0^{(m,m')}(z) \langle \bar{\partial}\hat{\phi}(\bar{z}) \exp(\mathrm{i}\hat{\phi}(w_1,\bar{w}_1)/2) \exp(-\mathrm{i}\hat{\phi}(w_2,\bar{w}_2)/2)\rangle \\
&\qquad+ \bar{\partial}\phi_0^{(m,m')}(\bar{z}) \langle \partial\hat{\phi}(z) \exp(\mathrm{i}\hat{\phi}(w_1,\bar{w}_1)/2) \exp(-\mathrm{i}\hat{\phi}(w_2,\bar{w}_2)/2)\rangle \\
&\qquad+ \langle \partial\hat{\phi}(z)\bar{\partial}\hat{\phi}(\bar{z}) \exp(\mathrm{i}\hat{\phi}(w_1,\bar{w}_1)/2) \exp(-\mathrm{i}\hat{\phi}(w_2,\bar{w}_2)/2)\rangle\Big].
\end{aligned} \tag{C.11}$$

The summation over m and m' can be eliminated by inserting the result of equation (C.10), together with some necessary differentiations, into equation (C.11):

$$\begin{aligned}
&\langle \partial\phi(z)\bar{\partial}\phi(\bar{z}) \exp(\mathrm{i}\phi(w_1,\bar{w}_1)/2) \exp(-\mathrm{i}\phi(w_2,\bar{w}_2)/2)\rangle_{\alpha=(1/2,1/2)} \\
&\quad= \frac{\sqrt{2\tau_2}}{Z_0} \exp[\pi(w_{12} - \bar{w}_{12})^2/8\tau_2][\langle \exp(\mathrm{i}\hat{\phi}(w_1,\bar{w}_1)/2) \exp(-\mathrm{i}\hat{\phi}(w_2,\bar{w}_2)/2)\rangle \\
&\qquad\times \left\{ -|\vartheta_1'(w_{12}/2)|^2 + \left(\frac{\pi}{\tau_2} + \frac{\pi^2(w_{12} - \bar{w}_{12})^2}{4\tau_2^2}\right) |\vartheta_1(w_{12}/2)|^2 \right. \\
&\qquad\left. + \frac{\pi(w_{12} - \bar{w_{12}})}{2\tau_2} \vartheta_1'(w_{12}/2)\bar{\vartheta}_1(\bar{w}_{12}/2) - \frac{\pi(w_{12} - \bar{w}_{12})}{2\tau_2} \vartheta_1(w_{12}/2)\bar{\vartheta}_1'(\bar{w}_{12}/2) \right\} \\
&\qquad- \mathrm{i}\langle \bar{\partial}\hat{\phi}(\bar{z}) \exp(\mathrm{i}\hat{\phi}(w_1,\bar{w}_1)/2) \exp(-\mathrm{i}\hat{\phi}(w_2,\bar{w}_2)/2)\rangle
\end{aligned}$$

$$\times \bar{\vartheta}_1(\bar{w}_{12}/2)\left(\vartheta_1'(w_{12}/2)+\frac{\pi(w_{12}-\bar{w}_{12})}{2\tau_2}\vartheta_1(w_{12}/2)\right)$$
$$-\,\mathrm{i}\langle\partial\hat{\phi}(z)\exp(\mathrm{i}\hat{\phi}(w_1,\bar{w}_1)/2)\exp(-\mathrm{i}\hat{\phi}(w_2,\bar{w}_2)/2)\rangle$$
$$\times \vartheta_1(w_{12}/2)\left(\bar{\vartheta}_1'(\bar{w}_{12}/2)-\frac{\pi(w_{12}-\bar{w}_{12})}{2\tau_2}\bar{\vartheta}_1(\bar{w}_{12}/2)\right)$$
$$+\langle\partial\hat{\phi}(z)\bar{\partial}\hat{\phi}(\bar{z})\exp(\mathrm{i}\hat{\phi}(w_1,\bar{w}_1)/2)\exp(-\mathrm{i}\hat{\phi}(w_2,\bar{w}_2)/2)\rangle|\vartheta_1(w_{12}/2)|^2]. \tag{C.12}$$

Only the correlators of the 'free part' of the boson remains to be determined. Its propagator is

$$\langle\hat{\phi}(z,\bar{z})\hat{\phi}(0,0)\rangle=-\ln\left|\frac{\vartheta_1(z)}{\vartheta_1'(0)}\right|^2-\frac{\pi(z-\bar{z})^2}{2\tau_2}. \tag{C.13}$$

Note that this propagator is the solution to the modified Green function satisfying

$$-\Delta G(z,\bar{z})=4\pi\delta^{(2)}(z)-\frac{4\pi}{\tau_2}. \tag{C.14}$$

From equation (C.13), one can obtain

$$\langle\exp(\mathrm{i}\hat{\phi}(w_1,\bar{w}_1)/2)\exp(-\mathrm{i}\hat{\phi}(w_2,\bar{w}_2)/2)\rangle=\exp\left[-\frac{\pi(w_{12}-\bar{w}_{12})^2}{8\tau_2}\right]\left|\frac{\vartheta_1'(0)}{\vartheta_1(w_{12})}\right|^{1/2}. \tag{C.15}$$

Since

$$\begin{aligned}\langle\partial\hat{\phi}(z)\exp[\mathrm{i}\hat{\phi}(w_1,\bar{w}_1)/2]\exp[-\mathrm{i}\hat{\phi}(w_2,\bar{w}_2)/2]\rangle&=\exp\left[\frac{1}{4}\langle\hat{\phi}(w_1,\bar{w}_1)\hat{\phi}(w_2,\bar{w}_2)\rangle\right]\\&\quad\times\langle\partial\hat{\phi}(z)\exp[\mathrm{i}(\hat{\phi}(w_1,\bar{w}_1)-\hat{\phi}(w_2,\bar{w}_2))/2]\\&=\frac{\mathrm{i}}{2}\exp\left[\frac{1}{4}\langle\hat{\phi}(w_1,\bar{w}_1)\hat{\phi}(w_2,\bar{w}_2)\rangle\right]\\&\quad\times(\langle\partial\hat{\phi}(z)\hat{\phi}(w_1,\bar{w}_1)\rangle-\langle\partial\hat{\phi}(z)\hat{\phi}_c(w_2,\bar{w}_2)\rangle),\end{aligned} \tag{C.16}$$

equation (C.13) leads to

$$\begin{aligned}&\langle\partial\hat{\phi}(z)\exp[\mathrm{i}\hat{\phi}(w_1,\bar{w}_1)/2]\exp[-\mathrm{i}\hat{\phi}(w_2,\bar{w}_2)/2]\rangle\\&\quad=-\frac{\mathrm{i}}{2}\exp\left[-\frac{\pi(w_{12}-\bar{w}_{12})^2}{8\tau_2}\right]\left|\frac{\vartheta_1'(0)}{\vartheta_1(w_{12})}\right|^{1/2}\\&\quad\quad\times\left[\frac{\vartheta_1'(z-w_1)}{\vartheta_1(z-w_1)}-\frac{\vartheta_1'(z-w_2)}{\vartheta_1(z-w_2)}-\frac{\pi(w_{12}-\bar{w}_{12})}{\tau_2}\right].\end{aligned} \tag{C.17}$$

Similarly,

$$\begin{aligned}&\langle\bar{\partial}\hat{\phi}(\bar{z})\exp[\mathrm{i}\hat{\phi}(w_1,\bar{w}_1)/2]\exp[-\mathrm{i}\hat{\phi}(w_2,\bar{w}_2)/2]\rangle\\&\quad=-\frac{\mathrm{i}}{2}\exp\left[-\frac{\pi(w_{12}-\bar{w}_{12})^2}{8\tau_2}\right]\left|\frac{\vartheta_1'(0)}{\vartheta_1(w_{12})}\right|^{1/2}\\&\quad\quad\times\left[\frac{\bar{\vartheta}_1'(\bar{z}-\bar{w}_1)}{\bar{\vartheta}_1(\bar{z}-\bar{w}_1)}-\frac{\bar{\vartheta}_1'(\bar{z}-\bar{w}_2)}{\bar{\vartheta}_1(\bar{z}-\bar{w}_2)}+\frac{\pi(w_{12}-\bar{w}_{12})}{\tau_2}\right].\end{aligned} \tag{C.18}$$

Lastly,

$$\langle\partial\hat{\phi}(z)\bar{\partial}\hat{\phi}(\bar{z})\exp(\mathrm{i}\hat{\phi}(w_1,\bar{w}_1)/2)\exp(-\mathrm{i}\hat{\phi}(w_2,\bar{w}_2)/2)\rangle = -\frac{1}{4}\exp\left[-\frac{\pi(w_{12}-\bar{w}_{12})^2}{8\tau_2}\right] \times\left|\frac{\vartheta_1'(0)}{\vartheta_1(w_{12})}\right|^{1/2}\left|\frac{\vartheta_1'(z-w_1)}{\vartheta_1(z-w_1)}-\frac{\vartheta_1'(z-w_2)}{\vartheta_1(z-w_2)}-\frac{\pi(w_{12}-\bar{w}_{12})}{\tau_2}\right|^2 -\frac{\pi}{\tau_2}\exp\left[-\frac{\pi(w_{12}-\bar{w}_{12})^2}{8\tau_2}\right]\left|\frac{\vartheta_1'(0)}{\vartheta_1(w_{12})}\right|^{1/2}. \quad (C.19)$$

The second term of this equation comes from the fact that from equation (C.13), $\langle\partial\hat{\phi}(z)\bar{\partial}\hat{\phi}(\bar{z})\rangle$ is actually nonzero.

Inserting equations (C.15), (C.17), (C.18) and (C.19) into equation (C.12) gives

$$\langle\partial\phi(z)\bar{\partial}\phi(\bar{z})\exp(\mathrm{i}\phi(w_1,\bar{w}_1)/2)\exp(-\mathrm{i}\phi(w_2,\bar{w}_2)/2)\rangle_{\alpha=(1/2,1/2)} = -\frac{\sqrt{2\tau_2}}{Z_0}\left|\frac{\vartheta_1'(0)}{\vartheta_1(w_{12})}\right|^{1/2} \times\left|\vartheta_1'(w_{12}/2)+\frac{1}{2}\vartheta_1(w_{12}/2)\left(\frac{\vartheta_1'(z-w_1)}{\vartheta_1(z-w_1)}-\frac{\vartheta_1'(z-w_2)}{\vartheta_1(z-w_2)}\right)\right|^2. \quad (C.20)$$

Similarly,

$$\langle\partial\phi(z)\bar{\partial}\phi(\bar{z})\exp(-\mathrm{i}\phi(w_1,\bar{w}_1)/2)\exp(\mathrm{i}\phi(w_2,\bar{w}_2)/2)\rangle_{\alpha=(1/2,1/2)} = -\frac{\sqrt{2\tau_2}}{Z_0}\left|\frac{\vartheta_1'(0)}{\vartheta_1(w_{12})}\right|^{1/2} \times\left|\vartheta_1'(-w_{12}/2)-\frac{1}{2}\vartheta_1(-w_{12}/2)\left(\frac{\vartheta_1'(z-w_1)}{\vartheta_1(z-w_1)}-\frac{\vartheta_1'(z-w_2)}{\vartheta_1(z-w_2)}\right)\right|^2. \quad (C.21)$$

Equations (C.20) and (C.21), together with equations (C.2) and (C.3), lead to the following result stated in section 3.2:

$$\langle\psi(z)\sigma(w_1)\sigma(w_2)\rangle_{\alpha=(1/2,1/2)} \propto \left[\frac{1}{\vartheta_1(w_{12})}\right]^{1/8} \times\left[\vartheta_1'(w_{12}/2)+\frac{1}{2}\vartheta_1(w_{12}/2)\left(\frac{\vartheta_1'(z-w_1)}{\vartheta_1(z-w_1)}-\frac{\vartheta_1'(z-w_2)}{\vartheta_1(z-w_2)}\right)\right]^{1/2}. \quad (C.22)$$

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